First-order Sobolev spaces, self-similar energies and energy measures on the Sierpiński carpet

Mathav Murugan[*] Ryosuke Shimizu[†]

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Abstract

For any $p \in (1, \infty)$, we construct p-energies and the corresponding p-energy measures on the Sierpiński carpet. A salient feature of our Sobolev space is the self-similarity of energy. An important motivation for the construction of self-similar energy and energy measures is to determine whether or not the Ahlfors regular conformal dimension is attained on the Sierpiński carpet. If the Ahlfors regular conformal dimension is attained, we show that any optimal Ahlfors regular measure attaining the Ahlfors regular conformal dimension must necessarily be a bounded perturbation of the p-energy measure of some function in our Sobolev space, where p is the Ahlfors regular conformal dimension. Under the attainment of the Ahlfors regular conformal dimension, the $(1, p)$-Newtonian Sobolev space corresponding to any optimal Ahlfors regular metric and measure is shown to coincide with our Sobolev space with comparable norms, where p is the Ahlfors regular conformal dimension.

Keywords: Sierpiński carpet, Sobolev space, Ahlfors regular conformal dimension, Loewner space

Contents

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1 Introduction and main results

The goal of this work is to construct and investigate properties of $(1,p)$-Sobolev space, p-energy and p-energy measures on the Sierpiński carpet. Our $(1,p)$-Sobolev space can be considered to be an analogue of $W^{1,p}(\mathbb{R}^n)$ on Euclidean space, the p-energy of a function f is an analogue of $\int_{\mathbb{R}^n} |\nabla f|^p(x)\,dx$, and the p-energy measure of a function f is an analogue of the measure $A \mapsto \int_A |\nabla f|^p(x)\,dx$. Similar $(1,p)$-Sobolev spaces were constructed in recent works of Kigami and the second-named author but much of the results there only apply to the case $p > \dim_{\mathrm{ARC}}$, where $\dim_{\mathrm{ARC}}$ is the Ahlfors regular conformal dimension [Shi24, Kig23].

Our approach and that of [Shi24, Kig23] goes back to the analytic construction of Brownian motion on the Sierpiński carpet by Kusuoka and Zhou [KZ92]. (The first construction of Brownian motion on the Sierpiński carpet was done by Barlow and Bass [BB89] in a purely probabilistic way.) The Dirichlet form corresponding to the Brownian motion on the Sierpiński carpet is a special case of p-energy when $p = 2$. The idea behind defining a p-energy of a function f on a metric space (X,d) is to approximate a metric space by a sequence of graphs $\{\mathbb{G}_n = (V_n, E_n) : n \in \mathbb{N}\}$ on a sequence of increasingly finer scales and to consider a sequence of discrete approximations $M_n f : V_n \to \mathbb{R}$ of the function $f : X \to \mathbb{R}$. Consider the *discrete p-energies*,

$$\mathcal{E}_p^{\mathbb{G}_n}(M_n f) = \sum_{\{x,y\} \in E_n} |(M_n f)(x) - (M_n f)(y)|^p.$$

We then choose a sequence $\{r_n : n \in \mathbb{N}\}$ of re-scaling factors $r_n \in (0,\infty)$ so that the quantities $\limsup_{n\to\infty} r_n \mathcal{E}_p^{\mathbb{G}_n}(M_n f), \liminf_{n\to\infty} r_n \mathcal{E}_p^{\mathbb{G}_n}(M_n f)$, and $\sup_{n\in\mathbb{N}} r_n \mathcal{E}_p^{\mathbb{G}_n}(M_n f)$ are comparable uniformly for all integrable functions f. The existence of such a sequence r_n is guaranteed by analytic properties on the sequence of graphs $\mathbb{G}_n$ such as bounds on capacity and Poincaré inequality. The Sobolev space is then defined as

$$\mathcal{F}_p := \left\{ f \in L^p \;\middle|\; \sup_{n\in\mathbb{N}} r_n \mathcal{E}_p^{\mathbb{G}_n}(M_n f) < \infty \right\}.$$

To describe our results, we prepare some notations of the Sierpiński carpet (see Definition 8.1 for details) and its approximations. The Sierpiński carpet K is the invariant set of eight contraction maps $\{F_i\}_{i\in\{1,\dots,8\}}$, where each contraction F_i maps the square $[-1,1]^2$

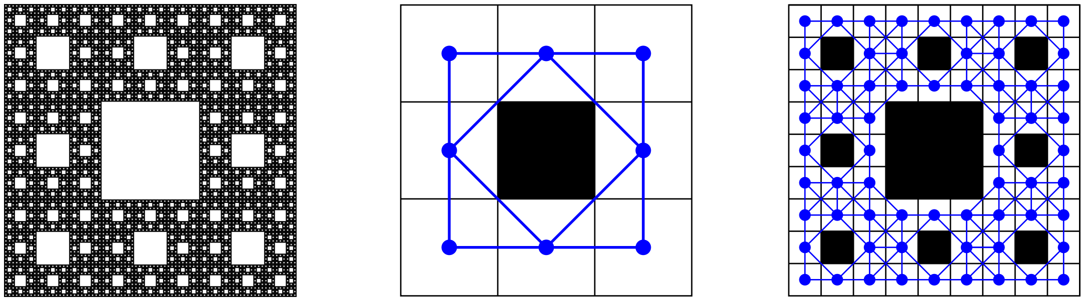

Figure 1.1: The planar Sierpiński carpet and its approximation graphs $\{\mathbb{G}_n\}$. ($\mathbb{G}_1$ and $\mathbb{G}_2$ are drawn in blue.)

to one of nine divided identical squares with side length 2/3 except for the central one, i.e. $F_i(x) = 3^{-1}(x - q_i) + q_i$ for some $q_i \in \mathbb{R}^2$ and $K = \bigcup_{i \in \{1,\dots,8\}} F_i(K)$. Let $V_n = S^n$ denote the set of words of length n over the alphabet $S = \{1, 2, \dots, 8\}$. For $w = w_1 \cdots w_n \in V_n$, we set $F_w \coloneqq F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_n}$. Let $\mathbb{G}_n = (V_n, E_n)$ be the graph whose vertex set is the set of words V_n with n-alphabets and the edge set is defined by

$$E_n = \{\{u, v\} : u, v \in V_n, F_u(K) \cap F_v(K) \neq \emptyset\}.$$

The sequence of graphs $\mathbb{G}_n, n \in \mathbb{N}$ approximate the Sierpiński carpet K (see Figure 1.1).

We now describe how to approximate a function on K by a function on $\mathbb{G}_n$. To this end, we equip K with the Euclidean metric d and the self-similar Borel probability measure m on K such that $m(F_w(K)) = 8^{-n}$ for all $w \in V_n, n \in \mathbb{N}$. For $n \in \mathbb{N}$, we define the discrete approximation operators $M_n : L^p(K, m) \to \mathbb{R}^{V_n}$ as

$$(M_n f)(u) \coloneqq \frac{1}{m(F_u(K))} \int_{F_u(K)} f \, dm, \quad \text{for all } u \in V_n.$$

For any $p \in (1, \infty)$, we show the existence of an exponent $\rho(p) \in (0, \infty)$ and some constant $C \in (1, \infty)$ such that

$$\sup_{n \in \mathbb{N}} \rho(p)^n \mathcal{E}_p^{\mathbb{G}_n}(M_n f) \leq C \limsup_{n \to \infty} \rho(p)^n \mathcal{E}_p^{\mathbb{G}_n}(M_n f) \leq C^2 \liminf_{n \to \infty} \rho(p)^n \mathcal{E}_p^{\mathbb{G}_n}(M_n f)$$

for all $f \in L^p(K, m)$. This implies that each of the three expressions in the above display are uniformly comparable up to multiplicative constants. One of them, say $\sup_{n \in \mathbb{N}} \rho(p)^n \mathcal{E}_p^{\mathbb{G}_n}(M_n f)$ could be a considered as a candidate p-energy. However, we would like to construct an *improved* p-energy $\mathcal{E}_p : \mathcal{F}_p \to [0, \infty)$ that is comparable to the above candidate p-energy but satisfies desirable properties such as self-similarity, Lipschitz contractivity, and strong locality that the above candidate need not satisfy. The definitions of these properties are included in the statement of Theorem 1.1. For $f \in L^p(K, m)$, by $\mathrm{supp}_m[f]$ we denote the support of the measure $f \, dm$. The following theorem describes the definition and basic properties of our Sobolev spaces.

Theorem 1.1 (Construction of $(1, p)$-Sobolev space and p-energy)**.** *Let $p \in (1, \infty)$ and let (K, d, m) be the Sierpiński carpet equipped with the Euclidean metric and the self-similar* *measure described above. Then there exists $\rho(p) \in (0,\infty)$ such that the normed linear space $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ defined by*

$$\mathcal{F}_p := \left\{ f \in L^p(K,m) \,\middle|\, \int_K |f|^p\,dm + \sup_{n\in\mathbb{N}} \rho(p)^n \mathcal{E}_p^{\mathbb{G}_n}(M_n f) < \infty \right\},$$

and

$$|f|_{\mathcal{F}_p} := \left(\sup_{n\in\mathbb{N}} \rho(p)^n \mathcal{E}_p^{\mathbb{G}_n}(M_n f) \right)^{1/p}, \quad \|f\|_{\mathcal{F}_p} := \|f\|_{L^p(m)} + |f|_{\mathcal{F}_p},$$

satisfies the following properties.

(i) $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ *is a reflexive separable Banach space.*

(ii) (Regularity) $\mathcal{F}_p \cap \mathcal{C}(K)$ *is a dense subspace in the Banach spaces* $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ *and* $(\mathcal{C}(K), \|\cdot\|_\infty)$.

Furthermore, there exist $C \geq 1$ and $\mathcal{E}_p \colon \mathcal{F}_p \to [0,\infty)$ satisfying the following:

(iii) $\mathcal{E}_p(\,\cdot\,)^{1/p}$ *is a semi-norm satisfying* $C^{-1}|f|_{\mathcal{F}_p} \leq \mathcal{E}_p(f)^{1/p} \leq C|f|_{\mathcal{F}_p}$ *for all* $f \in \mathcal{F}_p$.

(iv) (Uniform convexity) $\mathcal{E}_p(\,\cdot\,)^{1/p}$ *is uniformly convex.*

(v) (Lipschitz contractivity) *For any $f \in \mathcal{F}_p$ and 1-Lipschitz map $\varphi \in \mathcal{C}(\mathbb{R})$, we have $\varphi \circ f \in \mathcal{F}_p$ and $\mathcal{E}_p(\varphi\circ f) \leq \mathcal{E}_p(f)$.*

(vi) (Spectral gap) *It holds that*

$$\|f - f_K\|_{L^p(m)}^p \leq C\mathcal{E}_p(f) \quad \text{for all } f \in \mathcal{F}_p,$$

where $f_K := \fint_K f\,dm$ is the m-average of f. In particular,

$$\{f \in \mathcal{F}_p : \mathcal{E}_p(f) = 0\} = \{f \in L^p(K,m) : f \text{ is constant } m\text{-a.e.}\}. \tag{1.1}$$

(vii) (Strong locality) *If $f, g \in \mathcal{F}_p$ satisfy $\operatorname{supp}_m[f] \cap \operatorname{supp}_m[g - a\mathbb{1}_K] = \emptyset$ for some $a \in \mathbb{R}$, then $\mathcal{E}_p(f+g) = \mathcal{E}_p(f) + \mathcal{E}_p(g)$.*

(viii) (Self-similarity) *For any $f \in \mathcal{F}_p$, we have $f \circ F_i \in \mathcal{F}_p$ for all $i \in S$ and*

$$\mathcal{E}_p(f) = \rho(p) \sum_{i\in S} \mathcal{E}_p(f \circ F_i). \tag{1.2}$$

Furthermore, $\mathcal{F}_p \cap \mathcal{C}(K) = \{f \in \mathcal{C}(K) \mid f \circ F_i \in \mathcal{F}_p \text{ for all } i \in S\}$.

(ix) (Symmetry) *Let D_4 denote the dihedral group of isometries of K. For any $f \in \mathcal{F}_p$ and $\Phi \in D_4$, we have $f \circ \Phi \in \mathcal{F}_p$ and $\mathcal{E}_p(f \circ \Phi) = \mathcal{E}_p(f)$.*

Remark. One can show the generalized p-contraction property, which was introduced in [KS24+b] after submitting this work, for $(\mathcal{E}_p, \mathcal{F}_p)$. See also [KS24+b, Remark 8.20]. This property gives improvements of Theorem 1.1(iv),(v).

We compare the above result with earlier results in [Shi24, Kig23]. Theorem 1.1 was previously known only in the case $p > \dim_{\mathrm{ARC}}(K,d)$, where $\dim_{\mathrm{ARC}}(K,d) \in (1,\infty)$ is the Ahlfors regular conformal dimension [Shi24] (we recall the definition of Ahlfors regular conformal dimension in Definition 1.6). Similar to this work, Kigami uses an approach based on discrete energies and introduces a *conductive homogeneity* condition under which the Sobolev space was constructed [Kig23]. However much of the results apply only to the case $p > \dim_{\mathrm{ARC}}(K,d)$ as the author points out "Regrettably, we do not have much for the case $p \le \dim_{\mathrm{ARC}}(K,d)$" in [Kig23, p. 8]. In particular, Theorem 1.1 answers a question of Kigami [Kig23, §6.3, Problem 1] for the Sierpiński carpet which asks for the property (ii) above. This property is known as *regularity* in the theory of Dirichlet form [FOT, p. 6].

The difficulty in the case $p \le \dim_{\mathrm{ARC}}(K,d)$ is due to the fact that the Sobolev space contains discontinuous functions. Indeed, by a recent result by Cao, Chen and Kumagai [CCK24, Theorem 1.1], under the conductive homogeneity condition, the Sobolev space constructed by Kigami [Kig23] contains discontinuous functions if and only if $p \le \dim_{\mathrm{ARC}}(K,d)$. If $p > \dim_{\mathrm{ARC}}(K,d)$, there is a version of Morrey's embedding theorem which makes the analysis easier.

Another difficulty is that the conductive homogeneity condition of [Kig23] (or its analogue 'knight move condition' in [Shi24]) was not obtained on the Sierpiński carpet if $p \le \dim_{\mathrm{ARC}}(K,d)$. The Poincaré inequality for graphs $\mathbb{G}_n$ shown in our work (Theorem 4.2) implies these conditions when $p \le \dim_{\mathrm{ARC}}(K,d)$ for the Sierpiński carpet. We do not show them because their proofs are technical. (A proof is available in the long version [MS, Appendix C].) Our approach only relies on Poincaré inequality and certain upper bounds on capacity across annulus on the sequence of graphs $\mathbb{G}_n$.

As we will see in Theorem 1.4, the value of $\rho(p)$ in Theorem 1.1 is uniquely determined by the above properties. If $\rho(p)$ were larger, the Sobolev space $\mathcal{F}_p$ would only consist of constant functions violating property (ii). If $\rho(p)$ were smaller, then the resulting p-energy would be too small to satisfy property (vi).

Our next result is the existence of energy measures. To motivate energy measure, let us consider the following question: *what information does the energy measure contain about a function?* In the primary example on $\mathbb{R}^n$, the p-energy measure of a function $f \in W^{1,p}(\mathbb{R}^n)$ is the measure $A \mapsto \int_A |\nabla f(x)|^p \, dx$. By considering the Radon-Nikodym derivative of the energy measure with respect to Lebesgue measure, we see that the energy measure contains the same information as $|\nabla f|$ up to sets of Lebesgue measure zero, were ∇f is the distributional gradient of f. A generalization of $|\nabla f|$ is given by the *minimal p-weak upper gradient* in the theory of Newton-Sobolev space [HKST]. In these settings, the energy measure is always absolutely continuous with respect to the reference measure. In the setting of diffusion on fractals, the energy measure (for $p = 2$) is typically singular with respect to the reference measure [Hin05, KM20]. As we will see in Theorem 1.7, not requiring the p-energy measure to be absolutely continuous with respect to the reference measure is useful as the reference measure might not be suited to express energies and also because the energy measure might satisfy better properties such as the Loewner property. Based on the above analogy, we think of our energy measures as containing similar information about the function as the minimal p-weak upper gradient in the setting of Newton-Sobolev spaces.

Let us describe the construction of energy measure. Following an idea in [Hin05, Kus89], we use the self-similarity property of the p-energy to construct our p-energy measure. To describe it, we let $\Sigma = S^{\mathbb{N}}$ be the set of all infinite words in the alphabet S equipped with the product topology. Recall that the *canonical projection* $\chi\colon \Sigma \to K$ is defined to satisfy $\{\chi(\omega)\} = \bigcap_{n\in\mathbb{N}}(F_{w_1} \circ \cdots \circ F_{w_n})(K)$ for any $\omega = (w_1, w_2, \cdots) \in \Sigma$. For $w \in S^n$, let $\Sigma_w \subset \Sigma$ be the set of infinite words whose beginning n alphabets coincide with w. For any function $f \in \mathcal{F}_p$, self-similarity of the p-energy $\mathcal{E}_p(\cdot)$ and Kolmogorov's extension theorem guarantees the existence of a measure $\mathfrak{m}_p\langle f\rangle$ on Σ such that $\mathfrak{m}_p\langle f\rangle(\Sigma_w) = \rho(p)^n\mathcal{E}_p(f \circ F_w)$ for all $w \in S^n, n \in \mathbb{N}$. The energy measure is then defined to be the pushforward measure $\Gamma_p\langle f\rangle \coloneqq \chi_*(\mathfrak{m}_p\langle f\rangle)$. Our next theorem shows the existence of energy measure corresponding to self-similar energy and describes some of its basic properties.

Theorem 1.2 (Existence of p-energy measure)**.** *Let $p \in (1,\infty)$ and let (K, d, m) be the Sierpiński carpet. Let $(\mathcal{E}_p, \mathcal{F}_p)$ be the p-energy in Theorem 1.1. There exists a family of Borel finite measures $\{\Gamma_p\langle f\rangle\}_{f\in\mathcal{F}_p}$ on K satisfying the following:*

(i) *For any $f \in \mathcal{F}_p$, we have $\Gamma_p\langle f\rangle(K) = \mathcal{E}_p(f)$ and*

$$\Gamma_p\langle f\rangle(F_w(K)) = \rho(p)^n\mathcal{E}_p(f \circ F_w) \quad \text{for all } w \in S^n, n \in \mathbb{N}.$$

(ii) (Triangle inequality) *For any Borel set A of K, $\Gamma_p\langle\,\cdot\,\rangle(A)^{1/p}$ is a semi-norm on $\mathcal{F}_p$.*

(iii) (Lipschitz contractivity) *If $f \in \mathcal{F}_p$ and $\varphi \in \mathcal{C}(\mathbb{R})$ is 1-Lipschitz, then $\Gamma_p\langle\varphi\circ f\rangle(A) \le \Gamma_p\langle f\rangle(A)$ for any Borel set A of K.*

(iv) (Self-similarity) *For any $n \in \mathbb{N}$ and $f \in \mathcal{F}_p$,*

$$\Gamma_p\langle f\rangle = \rho(p)^n \sum_{w\in S^n}(F_w)_*\bigl(\Gamma_p\langle f \circ F_w\rangle\bigr).$$

(v) (Symmetry) *For any $f \in \mathcal{F}_p$ and $\Phi \in D_4$, we have $\Phi_*\bigl(\Gamma_p\langle f\rangle\bigr) = \Gamma_p\langle f \circ \Phi\rangle$.*

(vi) (Chain rule and strong locality) *For any $\Psi \in C^1(\mathbb{R})$ and $f \in \mathcal{F}_p \cap \mathcal{C}(K)$,*

$$\Gamma_p\langle\Psi \circ f\rangle(dx) = |\Psi'(f(x))|^p\Gamma_p\langle f\rangle(dx).$$

If $f, g \in \mathcal{F}_p \cap C(K)$ and $A \in \mathcal{B}(K)$ satisfy $(f-g)\big|_A = a \cdot \mathbb{1}_A$ for some $a \in \mathbb{R}$, then $\Gamma_p\langle f\rangle(A) = \Gamma_p\langle g\rangle(A)$.

Remark. After submitting this work to the journal, an improved version of the chain rule of self-similar p-energy measures is proved in [KS24+b]. Also, for each Borel set A of K, one can show that $(\Gamma_p\langle\,\cdot\,\rangle(A), \mathcal{F}_p)$ satisfies the generalized p-contraction property. See [KS24+b, Section 5] for details.

We describe another approach to defining Sobolev space motivated by a work of Korevaar and Schoen [KoSc]. This work describes classical Sobolev spaces in terms of Besov–Lipschitz spaces at the critical exponent (also called Korevaar–Schoen space). On a metric space (X,d), we denote by $B_{\mathsf{d}}(x,r)=\{y\in X : \mathsf{d}(x,y)<r\}$ the open ball centered at $x\in X$ and radius $r>0$. Our next result identifies our Sobolev space obtained using rescaled discrete energies in Theorem 1.1 as the critical Besov–Lipschitz or Korevaar–Schoen type space with comparable seminorms.

Definition 1.3. Let (X,d) be a connected metric space with $\#X\geq 2$ and let $\mathfrak{m}$ be a Borel-regular measure on X such that $\mathfrak{m}(B_{\mathsf{d}}(x,r))\in(0,\infty)$ for any $x\in X, r>0$. For $p\in(1,\infty)$ and $s>0$, the Besov–Lipschitz space $B^s_{p,\infty}=B^s_{p,\infty}(X,\mathsf{d},\mathfrak{m})$ is defined as

$$B^s_{p,\infty}\coloneqq\left\{f\in L^p(X,\mathfrak{m}) \;\middle|\; \sup_{r\in(0,2\operatorname{diam}(X,\mathsf{d}))}\int_X \fint_{B_{\mathsf{d}}(x,r)}\frac{|f(x)-f(y)|^p}{r^{sp}}\,\mathfrak{m}(dy)\mathfrak{m}(dx)<\infty\right\}.$$

Korevaar and Schoen show the coincidence $W^{1,p}(\mathbb{R}^n)=B^1_{p,\infty}(\mathbb{R}^n,d,\lambda)$ where d is the Euclidean metric and λ is the Lebesgue measure [KoSc, Theorem 1.6.2]. Furthermore there exists $C\in(0,\infty)$ such that the distributional gradient ∇f of any function $f\in W^{1,p}(\mathbb{R}^n)$ satisfies

$$C^{-1}\int_{\mathbb{R}^n}|\nabla f|^p\,d\lambda\leq\sup_{r\in(0,\infty)}\int_{\mathbb{R}^n}\fint_{B_d(x,r)}\frac{|f(x)-f(y)|^p}{r^p}\,\lambda(dy)\lambda(dx)\leq C\int_{\mathbb{R}^n}|\nabla f|^p\,d\lambda.$$

This result was later extended to spaces satisfying doubling property and Poincaré inequality by Koskela and MacManus [KoMa, Theorem 4.5]. In these settings, it turns out that the exponent $s=1$ is critical in that for every $s>1$ every function $f\in B^s_{p,\infty}$ is constant almost everywhere and for every $s\leq 1$, the space $B^s_{p,\infty}$ contains non-constant functions. This motivates the definition of the *critical exponent for Besov–Lipschitz space*

$$s_p\coloneqq\sup\{s>0 : B^s_{p,\infty}\text{ contains non-constant functions}\} \tag{1.3}$$

and the *Korevaar–Schoen space* as the *critical Besov–Lipschitz space* $B^{s_p}_{p,\infty}$. This approach to define Sobolev space was recently proposed by Baudoin [Bau24]. Our next result is that the Sobolev spaces defined using rescaled discrete energies coincides with the one defined using critical Besov–Lipschitz space with comparable seminorms. Furthermore, we describe the scaling constant $\rho(p)$ in Theorem 1.1 in terms of the critical scaling exponent for $B^s_{p,\infty}$.

Theorem 1.4 (Self-similar Sobolev space is a Korevaar–Schoen space)**.** *Let (K,d,m) be the Sierpiński carpet. Let $\mathcal{F}_p, |\cdot|_{\mathcal{F}_p}, \rho(p)$ be the Sobolev space, seminorm and scaling constant respectively as given in Theorem 1.1. Set $d_{\mathrm{w}}(p)\coloneqq\frac{\log(8\rho(p))}{\log 3}$ and*

$$J_{p,r}(f)\coloneqq\int_K\fint_{B_d(x,r)}|f(x)-f(y)|^p\,m(dy)m(dx)\quad\textit{for each } f\in L^p(K,m)\textit{ and } r>0.$$

Then, there exists $C\geq 1$ such that

$$C^{-1}|f|^p_{\mathcal{F}_p}\leq\liminf_{r\downarrow 0}r^{-d_{\mathrm{w}}(p)}J_{p,r}(f)\leq\sup_{r>0}r^{-d_{\mathrm{w}}(p)}J_{p,r}(f)\leq C\,|f|^p_{\mathcal{F}_p}\quad\textit{for all } f\in L^p(K,m),$$

and $d_{\mathrm{w}}(p)/p = s_p$*. In particular,* $\mathcal{F}_p(K,d,m) = B^{d_{\mathrm{w}}(p)/p}_{p,\infty}(K,d,m)$ *and*

$$\sup_{r>0} r^{-d_{\mathrm{w}}(p)} J_{p,r}(f) \leq C^2 \liminf_{r\downarrow 0} r^{-d_{\mathrm{w}}(p)} J_{p,r}(f) \quad \text{for all } f \in L^p(K,m). \tag{1.4}$$

This result was previously obtained under the additional assumption $p > \dim_{\mathrm{ARC}}(K,d)$. The above result answers a question of F. Baudoin as he asks if (1.4) is true for the Sierpiński carpet [Bau24]. Recently, Yang also proves (1.4) for generalized Sierpiński carpets in the case $p > \dim_{\mathrm{ARC}}$ [Yan23+, Theorem 2.8]. If (1.4) were true, then [Bau24] obtains number of useful consequences such as Sobolev embeddings and Gagliardo–Nirenberg inequalities. Our notation $d_{\mathrm{w}}(p)$ in Theorem 1.4 is inspired by the notion of walk dimension studied for $p = 2$ in the context of diffusion on fractals [KM23]. Similar to that setting, $d_{\mathrm{w}}(p)$ also plays a role as the exponent governing Poincaré inequality and capacity bounds as shown in the following theorem.

Theorem 1.5 (Poincaré inequality and capacity upper bound)**.** *Let* $p \in (1,\infty)$ *and let* (K,d,m) *be the Sierpiński carpet. Let* $\mathcal{E}_p, \mathcal{F}_p$ *be the p-energy and Sobolev space in Theorem 1.1. Let* $d_{\mathrm{w}}(p) = \frac{\log(8\rho(p))}{\log 3}$ *be as defined in Theorem 1.4 and let* $\Gamma_p\langle\,\cdot\,\rangle$ *denote the p-energy measure constructed in Theorem 1.2. Then there exist* $C, A \geq 1$ *such that for all* $x \in K$*,* $r > 0$ *and* $f \in \mathcal{F}_p$*, we have*

$$\int_{B_d(x,r)} \left| f - f_{B_d(x,r)} \right|^p dm \leq C r^{d_{\mathrm{w}}(p)} \int_{B_d(x,Ar)} d\Gamma_p\langle f \rangle,$$

and

$$\inf\left\{ \mathcal{E}_p(f) \;\middle|\; f \in \mathcal{F}_p \cap \mathcal{C}(K), f\big|_{B_d(x,r)} \equiv 1, \operatorname{supp}[f] \subseteq B_d(x,2r) \right\} \leq C \frac{m(B_d(x,r))}{r^{d_{\mathrm{w}}(p)}},$$

where $f_{B_d(x,r)} \coloneqq \frac{1}{m(B_d(x,r))} \int_{B_d(x,r)} f \, dm$*.*

Theorems 1.1, 1.4 and 1.5 suggest that the Sobolev space we construct is *canonical* since two different approaches lead to the same Sobolev space and natural analogies of Poincaré inequality and capacity upper bound hold in this framework. The most widely used definition of Sobolev space on a metric measure space relies on the notion of upper gradient introduced by Heinonen and Koskela [HK98]. Two different definitions of Sobolev space (the Newton-Sobolev space) based on upper gradient were proposed by Shanmugalingam [Sha00] and Cheeger [Che99] but these two definitions lead to the same Sobolev space on any metric measure space [HKST, Theorem 10.1.1]. The Newton-Soboev space $N^{1,p}(K,d,m)$ for the Sierpiński carpet is known to be trivial, that is, $N^{1,p}(K,d,m) = L^p(K,m)$ with equal norms, because the minimal weak upper gradient of any function is 0. We refer to Remark 9.6 for further details and references. The triviality of Sobolev space based on upper gradient suggests the need for an alternate method to construct Sobolev spaces on fractals such as the one considered in this work.

An important motivation for our work is quasisymmetric uniformization and the related attainment problem for Ahlfors regular conformal dimension. A recent work predicts that Sobolev spaces and energy measures are relevant to the attainment problem for Ahlfors regular conformal dimension [KM23, p.395-396]. Our work confirms this prediction. To describe our results in this direction, we recall the relevant definitions of conformal gauge and Ahlfors regular conformal dimension. Ahlfors regular conformal dimension is a slight variant of Pansu's conformal dimension [Pan] and first appeared in [BP03, BK05]. Conformal dimension of boundary of hyperbolic groups and Julia sets of complex dynamical systems are widely studied. We refer the reader to [MT] for a comprehensive account of conformal dimension.

Definition 1.6 (Conformal gauge)**.** Let (X,d) be a metric space and θ be another metric on X. We say that d is *quasisymmetric* to θ, if there exists a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that

$$\frac{\theta(x,y)}{\theta(x,z)} \le \eta\left(\frac{d(x,y)}{d(x,z)}\right) \quad \text{for all triples of points } x,y,z \in X,\ x \neq z.$$

The *conformal gauge* of a metric space (X,d) is defined as

$$\mathcal{J}(X,d) \coloneqq \{\theta : X \times X \to [0,\infty) \mid \theta \text{ is a metric on } X,\ d \text{ is quasisymmetric to } \theta\}. \tag{1.5}$$

A Borel measure μ on (X,d) is said to be *p-Ahlfors regular* with respect to d if there exists $C \ge 1$ such that

$$C^{-1} r^p \le \mu(B_d(x,r)) \le C r^p \quad \text{for all } x \in X, 0 < r < 2\operatorname{diam}(X,d).$$

The Ahlfors regular conformal dimension is defined as

$$\dim_{\mathrm{ARC}}(X,d) = \inf\left\{ p > 0 \;\middle|\; \begin{array}{l} \theta \in \mathcal{J}(X,d), \text{there is a Borel measure } \mu \text{ on } (X,\theta) \\ \text{that is } p\text{-Ahlfors regular with respect to } \theta \end{array} \right\}.$$

The infimum in the definition of $\dim_{\mathrm{ARC}}(X,d)$ need not be attained in general [BK05, §6]. The *attainment problem for Ahlfors regular conformal dimension* asks if the infimum in the definition of $\dim_{\mathrm{ARC}}(X,d)$ is attained by a 'optimal' metric and measure. *Quasisymmetric uniformization problem* asks if there is a metric in the conformal gauge isometric to a model space with more desirable properties. These two problems are often related. For instance, it is a well-known open problem to determine whether or not the conformal gauge of the standard Sierpiński carpet contains a Loewner metric [HKST, p. 408], [Kle06, Question 8.3] (we recall the definition of Loewner metric in Definition 9.11). Another related question is to determine if the Ahlfors regular conformal dimension of the Sierpiński carpet is attained [BK05, Problem 6.2]. As pointed out by Cheeger and Eriksson-Bique, these two questions are essentially the same due to the combinatorial Loewner property of the Sierpiński carpet [BK13, Theorem 4.1], [CE, §1.6].

As a motivation for the attainment problem for Ahlfors regular conformal dimension, we recall a long-standing conjecture in geometric group theory, namely Cannon's conjecture. It asserts that any Gromov hyperbolic group G whose boundary at infinity $\partial_\infty G$ is homeomorphic to $\mathbb{S}^2$ admits an action on the hyperbolic 3-space $\mathbb{H}^3$ that is isometric, properly discontinuous and cocompact. Bonk and Kleiner show Cannon's conjecture under the additional assumption that the Ahlfors regular conformal dimension of the boundary at infinity $\partial_\infty G$ is attained [BK05]. Thus Cannon's conjecture is reduced to an attainment problem for the Ahlfors regular conformal dimension of $\partial_\infty G$. We refer the reader to ICM 2006 proceedings of Bonk for further context and details [Bon06].

Another related motivation for the attainment problem for Ahlfors regular conformal dimension is to better understand Loewner spaces. Since Loewner spaces enjoy desirable properties, it is useful to know if a given metric space contains a Loewner metric in its conformal gauge. To this end, Kleiner formulated a combinatorial version of Loewner property that is necessary for such a Loewner metric to exist and is easier to check. Bourdon and Kleiner verify combinatorial Loewner property for a number of examples including the Sierpiński carpet [BK13]. Kleiner conjectured that the combinatorial Loewner property for a self-similar space is equivalent to the existence of Loewner metric in the conformal gauge [Kle06, Conjecture 7.5]. Due to an observation of Cheeger and Eriksson-Bique [CE, §1.6], Kleiner's conjecture can be rephrased as a conjecture about the attainment problem as follows: combinatorial Loewner property for a self-similar space implies that the Ahlfors regular conformal dimension is attained. We refer to the ICM 2006 proceedings of Kleiner for further details and background [Kle06]. Added in revision: There has been a very recent progress on Kleiner's conjecture [Kle06, Conjecture 7.5]. Anttila and Eriksson-Bique [AEB24+a] have verified that some variants of Laakso spaces [Laa00, Laa02] give counterexamples to Kleiner's conjecture, i.e., these spaces do not contain any Loewner metric in their conformal gauges while these satisfy the combinatorial Loewner property.

In contrast to the attainment problem, the value of the Ahlfors regular conformal dimension is better understood. Indeed, Keith and Kleiner [BK13, Remark on p. 63] showed that the Ahlfors regular conformal dimension can be described as a critical exponent of discrete energies (or equivalently, discrete modulus). Carrasco gave an independent proof of this fact [Car, Theorem 1.3] (see also [Kig20, Mur23b, Sha23] for related works). Since we can understand the value of Ahlfors regular conformal dimension using discrete energies, it is also natural to try to understand the attainment problem using discrete energies. Our next main result shows the relevance of discrete energies for the attainment problem on the Sierpiński carpet. We conjecture that similar results should be true on a large class of self-similar spaces.

As partial progress towards the attainment problem for Ahlfors regular conformal dimension on the Sierpiński carpet, we show that if an optimal measure attaining the Ahlfors regular conformal dimension exists then this measure is necessarily a bounded perturbation of the p-energy measure of some function in our $(1, p)$-Sobolev space, where p is the Ahlfors regular conformal dimension. This result confirms the relevance of energy measures to the attainment problem for Ahlfors regular conformal dimension as predicted earlier in [KM23, p.395-396]. Furthermore, if the Ahlfors regular conformal dimension is attained we identify our Sobolev space $\mathcal{F}_p$ with Newton-Sobolev space of the attaining metric measure space $N^{1,p}(X, \theta, \mu)$ (see Definition 9.5 for the precise definition). Moreover, the attaining measure is essentially equal to the energy measure $\Gamma_p\langle h\rangle$ for some function $h \in \mathcal{C}(K) \cap \mathcal{F}_p(K, d, m)$.

Theorem 1.7. *Let (K, d, m) denote the Sierpiński carpet and let $p = \dim_{\mathrm{ARC}}(K, d)$.*

Suppose that there exists $\theta \in \mathcal{J}(K,d)$ and a measure μ on K attaining the Ahlfors regular conformal dimension; that is, μ is a p-Ahlfors regular measure on (K,θ). Let $\mathcal{F}_p = \mathcal{F}_p(K,d,m), \mathcal{E}_p$ and $\Gamma_p\langle\,\cdot\,\rangle$ denote the Sobolev space, p-energy and p-energy measure as given in Theorem 1.2. Then we have the following:

(i) *The spaces $\mathcal{F}_p(K,d,m)$ and $N^{1,p}(K,\theta,\mu)$ are equal with comparable norms, semi-norms, and energy measure. More precisely, it holds that $\mathcal{C}(K)\cap\mathcal{F}_p(K,d,m) = \mathcal{C}(K)\cap N^{1,p}(K,\theta,\mu)$, there exist a bijective linear map $\iota\colon \mathcal{F}_p(K,d,m) \to N^{1,p}(K,\theta,\mu)$ and $C_1 > 1$ such that $\iota(f) = f$ for any $f \in \mathcal{C}(K)\cap\mathcal{F}_p(K,d,m) = \mathcal{C}(K)\cap N^{1,p}(K,\theta,\mu)$ (more precisely, the the equivalence class containing f in $\mathcal{F}_p(K,d,m)$ is mapped to the equivalence class containing f in $N^{1,p}(K,\theta,\mu)$) and*

$$C_1^{-1}\Gamma_p\langle f\rangle(B) \le \int_B g_{\iota(f)}^p\,d\mu \le C_1\Gamma_p\langle f\rangle(B)$$

for any Borel set $B \subset K, f \in \mathcal{F}_p(K,d,m)$, where $g_{\iota(f)}^p$ denotes the minimal p-weak upper gradient of $\iota(f)$. In particular, $C_1^{-1}\mathcal{E}_p(f) \le \int_K g_{\iota(f)}^p\,d\mu \le C_1\mathcal{E}_p(f)$ for all $f \in \mathcal{F}_p(K,d,m)$. Furthermore, the corresponding norms are comparable; that is,

$$C^{-1}\|f\|_{\mathcal{F}_p(K,d,m)} \le \|\iota(f)\|_{N^{1,p}(K,\theta,\mu)} \le C\|f\|_{\mathcal{F}_p(K,d,m)} \quad \textit{for all } f \in \mathcal{F}_p(K,d,m).$$

(ii) *There exist $h \in \mathcal{F}_p(K,d,m)\cap\mathcal{C}(K)$ and $C_2 \in (0,\infty)$ such that*

$$C_2^{-1}\Gamma_p\langle h\rangle(B) \le \mu(B) \le C_2\Gamma_p\langle h\rangle(B) \quad \textit{for any Borel set } B \subset K.$$

In particular, $\Gamma_p\langle h\rangle$ is a p-Ahlfors regular measure on (K,θ).

Let us briefly explain how Theorem 1.7 could be potentially used to solve the attainment problem. Although the attainment problem requires us to find optimal metrics and measures, it is well-known that the metrics and measures determine each other (see Lemmas 9.15 and 9.13). Therefore it suffices to look for optimal measure and use Lemma 9.13 to construct the corresponding metric. By Theorem 1.7, it suffices to look for optimal measures among energy measures of continuous functions. We conjecture that it suffices to look for optimal measure among energy measures of p-harmonic functions (see Conjecture 10.8). One could then hope to find a 'good' function whose energy measure is optimal or rule out the existence of such function by a careful analysis of energy measures. In fact, Theorem 1.7(ii) was inspired by a similar result for the attainment problem for conformal walk dimension [KM23, Theorem 6.16]. Such a result was successfully used to solve a similar attainment problem in [KM23].

More generally, we believe that Sobolev spaces and energy measures are relevant to similar quasisymmetric uniformization problems and the attainment problem for Ahlfors regular conformal dimension on other 'self-similar spaces' such as boundaries of hyperbolic groups and Julia sets in conformal dynamics. It would be interesting to construct Sobolev space, energy measures and prove analogues of Theorem 1.7 for fractals arising from hyperbolic groups and conformal dynamics [Bon06, Kle06]. Another obvious question is to use Theorem 1.7 to solve the attainment problem. This motivates further study of energy measures and p-harmonic functions.

Although we discussed three approaches towards defining Sobolev space based on discrete energies, Korevaar–Schoen energies, and upper gradients, there are several omissions. Among them, we mention Sobolev spaces constructed using two-point estimates by Hajłasz (Hajłasz–Sobolev space) [Haj96], Poincaré inequalities by Hajłasz–Koskela (Poincaré-Sobolev space) [HK95, HK00], and using weak L^p-estimates of gradient on hyperbolic fillings by Bonk–Saksman [BS18]. It would be interesting to understand if these spaces or their variants are related to our Sobolev spaces constructed using discrete energies. Added in revision: There has been other recent works towards defining Sobolev spaces [BBR24, Kuw24]. Their definitions of Sobolev spaces are based on the modulus of gradient defined through a strongly local regular Dirichlet form.

1.1 Overview for the rest of the paper.

In §2, we introduce basic notions concerning capacity, modulus and volume growth of graphs.

In §3, we introduce variants of the ball Loewner property due to Bonk and Kleiner and of Loewner-type modulus lower bounds between connected sets. The main result (Theorem 3.2) shows that lower bounds of modulus between balls imply lower bounds of modulus between any pair of connected sets.

In §4, we use the lower bounds of modulus from §3 to obtain a discrete Poincaré inequality. The proof of the Poincaré inequality in Theorem 4.2 follows an idea of Heinonen and Koskela [HK98, Proof of Theorem 5.12].

In §5, we show that discrete Poincaré inequality along with capacity upper bounds on graphs imply elliptic Harnack inequality for p-harmonic functions on graphs. In this regime, to show the regularity of the Sobolev space we obtain Harnack inequality in a discrete setting by using a geometric argument in [Mur23a] which improves the results of [BCK05] in the case $p = 2$. The Harnack inequality is then used to prove existence of Hölder continuous cutoff functions with controlled energy.

In §6, we introduce a framework describing the approximation of a metric space by a sequence of graphs. We then define the Sobolev space using discrete graph energies under the assumption that the sequence of graphs satisfy uniform Poincaré inequality and capacity upper bounds. We obtain many basic properties of this Sobolev space such as completeness, separability, reflexivity, and the existence of a dense set of continuous functions in the Sobolev space.

In §7, we identify our Sobolev space as the Korevaar–Schoen space with comparable energies. We express the critial exponent for Besov–Lipschitz space in terms of the scaling exponent for discrete energies.

In §8, we apply the results from previous sections to the planar Sierpiński carpet. To this end, we check the assumptions imposed on the graph approximations for the construction of the Sobolev space in §6 and *pre-self-similarity condition* imposed to construct a self-similar p-energy. We also describe the construction of the energy measure associated to a self-similar p-energy and obtain its basic properties.

In §9, we show that any optimal measure for Ahlfors regular conformal dimension on the Sierpiński carpet must necessarily be comparable to a energy measure. If the Ahlfors regular conformal dimension is attained we identify the Newton-Sobolev space of the attaining space with our Sobolev space.

In §10, we collect some conjectures and open problems related to our work.

Many of the proofs that rely on small modifications to known methods have been omitted but an interested reader can find more complete proofs on arXiv [MS].

Notations. In this paper, we use the following notation and conventions.

(1) $\mathbb{N} := \{n \in \mathbb{Z} \mid n > 0\}$ and $\mathbb{Z}_{\geq 0} := \mathbb{N} \cup \{0\}$.

(2) For a set A, we write $\#A$ to denote the cardinality of A.

(3) Let X be a non-empty set. For disjoint subsets A and B of X, we use $A \sqcup B$ to denote the disjoint union of A and B.

(4) For $a \in \mathbb{R}$, define $\mathrm{sgn}(a) := \mathbb{1}_{(0,\infty)}(a) - \mathbb{1}_{(-\infty,0)}(a)$.

(5) For $a, b \in \mathbb{R}$, we write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. Set $a^+ = a \vee 0$ and $a^- = a \wedge 0$ for any $a \in \mathbb{R}$. We also use these notations for real-valued functions.

(6) For $a \in \mathbb{R}$, define $\lceil a \rceil, \lfloor a \rfloor \in \mathbb{Z}$ by

$$\lceil a \rceil = \max\{n \in \mathbb{Z} \mid n \leq a\} \quad \text{and} \quad \lfloor a \rfloor = \min\{n \in \mathbb{Z} \mid a \leq n\}.$$

(7) For arbitrary countable set V, define $\mathbb{R}^V = \{f \mid f\colon V \to \mathbb{R}\}$, $\ell^+(V) = [0,+\infty)^V = \{f \mid f\colon V \to [0,+\infty)\}$ and $\ell^+_c(V) = \{f \in [0,+\infty)^V \mid \#\,\mathrm{supp}[f] < +\infty\}$, where $\mathrm{supp}[f] := \{x \in V \mid f(x) \neq \emptyset\}$. Also, set $\mathrm{osc}_V[f] := \sup_{x,y\in V} |f(x) - f(y)|$ for $f \in \mathbb{R}^V$.

(8) Let (X, d) be a metric space. The open ball with center $x \in X$ and radius $r > 0$ is denoted by $B_d(x, r)$, that is, $B_d(x, r) := \{y \in X \mid d(x, y) < r\}$. If the metric d is clear in context, then we write $B(x, r)$ for short. We write $\overline{B}(x, R)$ for $\{y \in X \mid d(x, y) \leq r\}$. For a metric ball B, let $\mathrm{rad}(B)$ denote the radius of B. For $\lambda \geq 0$ and a ball $B = B(x, R)$, define $\lambda B = B(x, \lambda R)$.

(9) Let (X, d) be a metric space. We define $\mathrm{diam}(A, d) := \sup_{x,y\in A} d(x, y)$ and $\mathrm{dist}_d(A, B) := \inf\{d(x, y) \mid x \in A, y \in B\}$ for $A, B \subseteq X$. We also use $\mathrm{diam}_d(A)$ to denote $\mathrm{diam}(A, d)$. If no confusion can occur, we omit the metric d in these notations.

(10) Let $(X, \mathscr{A}, \mu)$ be a measure space. For $f \in L^1_{\mathrm{loc}}(X, \mu)$ and $A \in \mathscr{A}$ with $\mu(A) < +\infty$, we use $⨍_A f\,d\mu$ to denote the averaged integral of f over A, i.e.

$$⨍_A f\,d\mu = \frac{1}{\mu(A)} \int_A f(x)\,\mu(dx).$$

We also write f_A or $(f)_A$ to denote $⨍_A f\,d\mu$ if the underlying measure μ is clear.

(11) Let $(X, \mathscr{A}, \mu)$ be a measure space and let $1 \leq p \leq \infty$. For $f \in L^p(X, \mu)$, we use $\|f\|_p$ to denote the L^p-norm of f. In addition, for any $A \in \mathscr{A}$, define

$$\|f\|_{p,A} := \|f \mathbb{1}_A\|_p = \left(\int_A |f(x)|^p\,\mu(dx) \right)^{1/p}.$$

(12) Let X be a topological space. We use $\mathscr{B}(X)$ (resp. $\mathscr{B}_+(X)$) to denote the set of $[-\infty,\infty]$-valued (resp. $[0,\infty]$-valued) Borel measurable functions on X. (Note that each element in $\mathscr{B}(X)$ or $\mathscr{B}_+(X)$ is defined on every points of X.)

2 Preliminaries

2.1 Basic facts and terminologies of graphs

Throughout this section, let $G = (V, E)$ be a locally finite connected simple non-directed graph, i.e. $G = (V, E)$ is a simple connected graph, where V is a countable set (the set of vertices) and $E \subseteq \bigl\{\{x, y\} \bigm| x, y \in V, x \neq y\bigr\}$ (the set of edges), satisfying

$$\deg_G(x) \coloneqq \#\{y \in V \mid \{x, y\} \in E\} < +\infty \quad \text{for all } x \in V.$$

We always consider G as a metric space equipped with the graph distance $d = d_G$. In this paper, we suppose that G has bounded degree, i.e. $\deg(G) \coloneqq \sup_{x\in V} \deg_G(x) < +\infty$.

A sequence of vertices $\theta = [x_0, \dots, x_n]$ for some $n \in \mathbb{N}$ is said to be a *(finite) path* in G if $x_i \in V$ and $\{x_i, x_{i+1}\} \in E$ for each $i \in \{0, \dots, n-1\}$. We frequently regard a path θ as a subset $\{x_i\}_{i=0}^n$ of V. Define the *length* of $\theta = [x_0, \dots, x_n]$ by $\mathsf{len}_G(\theta) \coloneqq n$. A finite path $\theta = [x_0, \dots, x_n]$ is said to be *simple* if there is no loops, i.e. $x_i \neq x_j$ for any distinct $i, j \in \{0, \dots, n\}$. Note that our definition excludes the case where a one point set $\{x\}$ becomes a path (since G has no self-loops). In particular, $\mathsf{len}(\theta) \in \mathbb{N}$ for a finite path θ.

For any subset $A \subseteq V$, we define

$$E(A) \coloneqq \bigl\{\{x, y\} \in E \bigm| x, y \in A\bigr\}.$$

A subset $A \subseteq V$ is called a *connected subset of V (with respect to G)* if $d_{(A,E(A))}(x, y) < \infty$ for all $x, y \in A$.

For arbitrary $A \subseteq V$, define

$$\partial_i A = \{x \in A \mid \text{there exists } y \in V \setminus A \text{ such that } \{x, y\} \in E\},$$

$$\partial A = \{x \in V \setminus A \mid \text{there exists } y \in A \text{ such that } \{x, y\} \in E\},$$

and $\overline{A} = A \cup \partial A$. The set $\partial_i A$ (resp. ∂A) is called the interior (resp. exterior) boundary of A in G. The set $\overline{A}$ is a kind of closure of A in G.

2.2 Combinatorial p-modulus of path families

We recall the notion of combinatorial modulus of discrete path families on a graph and a few basic properties. For a path θ in $G = (V, E)$ and $\rho \in \ell^+(V)$, define the *ρ-length of θ, $L_\rho(\theta)$,* by

$$L_\rho(\theta) = \sum_{v\in\theta} \rho(v).$$

For arbitrary path family Θ on G, define the *ρ-length of* Θ by $L_\rho(\Theta) := \inf_{\theta\in\Theta} L_\rho(\theta)$. The set of *admissible functions* $\mathsf{Adm}(\Theta)$ *for* Θ is given by

$$\mathsf{Adm}(\Theta) = \{\rho \in \ell^+(V) \mid L_\rho(\Theta) \geq 1\}.$$

Definition 2.1. Let Θ be a family of paths in G and let $p > 0$. The *(combinatorial) p-modulus* $\mathrm{Mod}_p^G(\Theta)$ *of* Θ is

$$\mathrm{Mod}_p^G(\Theta) = \inf_{\rho\in\mathsf{Adm}(\Theta)} \|\rho\|_{\ell^p(V)}^p = \inf_{\rho\in\mathsf{Adm}(\Theta)} \sum_{v\in V} \rho(v)^p.$$

We also use $\mathrm{Mod}_p(\Theta)$ to denote $\mathrm{Mod}_p^G(\Theta)$ when no confusion can occur.

Remark 2.2. For a path family Θ, define $V[\Theta] := \{v \in V \mid v \in \theta' \text{ for some } \theta' \in \Theta\}$. We easily see that $\rho \in \mathsf{Adm}(\Theta)$ implies $\rho\mathbb{1}_{V[\Theta]} \in \mathsf{Adm}(\Theta)$. This observation yields $\mathrm{Mod}_p^G(\Theta) = \inf_{\rho\in\mathsf{Adm}(\Theta)} \|\rho\|_{p,V[\Theta]}^p$.

The following properties of p-modulus is well-known.

Lemma 2.3 (e.g. [HKST, Section 5.2])**.** *Let* $p > 0$.

(i) $\mathrm{Mod}_p^G(\emptyset) = 0$.

(ii) *If path families* $\Theta_i\,(i = 1, 2)$ *satisfy* $\Theta_1 \subseteq \Theta_2$, *then* $\mathrm{Mod}_p^G(\Theta_1) \subseteq \mathrm{Mod}_p^G(\Theta_2)$.

(iii) *For any sequence of path families* $\{\Theta_n\}_{n\in\mathbb{N}}$,

$$\mathrm{Mod}_p^G\left(\bigcup_{n\in\mathbb{N}} \Theta_n\right) \leq \sum_{n=1}^{\infty} \mathrm{Mod}_p^G(\Theta_n).$$

(iv) *Let* $\Theta, \Theta_\#$ *be families of paths. If all path* $\theta \in \Theta$ *has a sub-path* $\theta_\# \in \Theta_\#$ *(i.e.* $\theta_\# \subseteq \theta$*), then* $\mathrm{Mod}_p^G(\Theta) \leq \mathrm{Mod}_p^G(\Theta_\#)$.

If $p > 1$, then by the strict convexity of ℓ^p, there exists a unique $\rho \in \mathsf{Adm}(\Theta)$ such that $\mathrm{Mod}_p^G(\Theta) = \sum_{v\in V} \rho(v)^p$.

For subsets $A_i \subseteq V\ (i = 0, 1, 2)$ with $A_0 \cup A_1 \subseteq A_2$, define

$$\mathsf{Path}(A_0, A_1; A_2) = \left\{ [x_0, \dots, x_n] \;\middle|\; \begin{array}{c} n \in \mathbb{N}, \{x_i, x_{i+1}\} \in E \text{ for any } i = 0, \dots, n-1, \\ x_i \in A_2\,(i = 0, \dots, n), x_0 \in A_0, x_n \in A_1 \end{array} \right\},$$

and we write $\mathrm{Mod}_p(A_0, A_1; A_2)$ for $\mathrm{Mod}_p\bigl(\mathsf{Path}(A_0, A_1; A_2)\bigr)$. We use $\mathsf{Path}(A_0, A_1)$ and $\mathrm{Mod}_p(A_0, A_1)$ to denote $\mathsf{Path}(A_0, A_1; V)$ and $\mathrm{Mod}_p(A_0, A_1; V)$ respectively. If we need to specify the underlying graph G, we will use the notation $\mathsf{Path}_G(A_0, A_1; A_2)$.

The following lemma is used to obtain lower bounds on modulus. Roughly speaking, modulus lower bound of a curve family is equivalent to existence of shortcuts. This property is used in [BK05] and a direct consequence of the definition of modulus (and Hölder's inequality) as observed in [BK13, Lemma 2.7].

Lemma 2.4. *Let $p > 0$. Let Θ be a family of paths in G and let $c > 0$. If $\mathrm{Mod}_p(\Theta) \geq c$, then for any $\varepsilon > 0$ and $\rho \in \ell^+(V)$ there exists a path $\theta \in \Theta$ such that*

$$L_\rho(\theta) \leq (1+\varepsilon)c^{-1/p} \left\|\rho\right\|_{p,V[\Theta]}. \tag{2.1}$$

Conversely, if for any $\rho \in \ell^+(V)$ there exists a path $\theta \in \Theta$ such that $L_\rho(\theta) \leq c^{-1/p} \left\|\rho\right\|_p$, then $\mathrm{Mod}_p(\Theta) \geq c$. In particular, if $p \geq 1$, $L \in \mathbb{N}$ and there exists $\theta \in \Theta$ such that $\mathsf{len}(\theta) \leq L$, then

$$\mathrm{Mod}_p^G(\Theta) \geq L^{1-p}. \tag{2.2}$$

2.3 Discrete p-energy, p-Laplacian and associated capacity

For $f \in \mathbb{R}^V$, the *length of discrete gradient of f*, $|\nabla f| \colon E \to [0,+\infty)$, is given by

$$|\nabla f|\bigl(\{x,y\}\bigr) = |f(y) - f(x)| \quad \text{for } \{x,y\} \in E.$$

We abbreviate $|\nabla f|\bigl(\{x,y\}\bigr)$ as $|\nabla f|(x,y)$ for each $\{x,y\} \in E$.

Definition 2.5. Let $p > 0$ and let $A \subseteq V$. For $f, g \in \mathbb{R}^V$, define

$$\mathcal{E}_{p,A}^G(f;g) \coloneqq \sum_{\{x,y\}\in E(A)} \mathrm{sgn}(f(y)-f(x))|f(y)-f(x)|^{p-1}(g(y)-g(x)).$$

The p-energy of f on A is given by $\mathcal{E}_{p,A}^G(f) = \mathcal{E}_{p,A}^G(f;f)$, i.e.

$$\mathcal{E}_{p,A}^G(f) \coloneqq \sum_{\{x,y\}\in E(A)} |\nabla f|(x,y)^p = \sum_{\{x,y\}\in E(A)} |f(x)-f(y)|^p.$$

We write $\mathcal{E}_p^G(f;g)$ and $\mathcal{E}_p^G(f)$ for $\mathcal{E}_{p,V}^G(f;g)$ and $\mathcal{E}_{p,V}^G(f)$ respectively. We omit the underlying graph G in these notations if no confusion can occur.

We recall basic properties of discrete p-energy, which are immediate from the definition.

Lemma 2.6. *Let $p > 0$ and $A \subseteq V$.*

(a) $\mathcal{E}_{p,A}^G(f\wedge g) \vee \mathcal{E}_{p,A}^G(f\vee g) \leq \mathcal{E}_{p,A}^G(f) + \mathcal{E}_{p,A}^G(g)$ *for any $f, g \in \mathbb{R}^A$.*

(b) $\mathcal{E}_{p,A}^G(f\cdot g) \leq (2^{p-1}\vee 1)\bigl(\|g\|_{\ell^\infty(A)}^p \mathcal{E}_{p,A}^G(f) + \|f\|_{\ell^\infty(A)}^p \mathcal{E}_{p,A}^G(g)\bigr)$ *for any $f, g \in \mathbb{R}^A$.*

Next we recall the definition of discrete p-Laplacian using a discrete version of integration by parts. Let $\langle\cdot,\cdot\rangle_{\ell^2(V,\deg)}$ denote the inner product of $\ell^2(V,\deg)$

Definition 2.7. Let $p > 0$. The *p-Laplacian* $\Delta_p^G \colon \mathbb{R}^V \to \mathbb{R}^V$ on G is defined by, for $f \in \mathbb{R}^V$ and $x \in V$,

$$\bigl(\Delta_p^G f\bigr)(x) = \frac{1}{\deg(x)} \sum_{\substack{y\in V;\\(x,y)\in E}} \mathrm{sgn}(f(y)-f(x))|f(y)-f(x)|^{p-1}. \tag{2.3}$$

A function $f \in \mathbb{R}^V$ is said to be *p-superharmonic* (resp. *p-subharmonic*) at $x \in V$ if $\Delta_p^G f(x) \leq 0$ (resp. $\Delta_p^G f(x) \geq 0$). In addition, f is said to be *p-harmonic* at $x \in V$ if $\Delta_p^G f(x) = 0$. If $A \subseteq V$ and $\Delta_p^G f(x) = 0$ for every $x \in A$, then f is said to be p-harmonic in A. p-superharmonic, p-subharmonic functions in A are defined in similar ways.

The following lemma describes a well-known property of p-superharmonic (resp. p-subharmonic) functions, namely the *minimum (resp. maximum) principle*.

Lemma 2.8 ([HS97a, Theorem 3.14] or [MY92, Theorem 7.5])**.** *Let A be a non-empty connected subset of G. Let $f \in \mathbb{R}^V$ be p-superharmonic (resp. p-subharmonic) in A.*

(a) *If there exists $x \in A$ such that $f(x) = \min_{z \in \overline{A}} f(z)$ (resp. $f(x) = \max_{z \in \overline{A}} f(z)$), then f is constant on $\overline{A}$.*

(b) *If A is finite, then $\min_{\partial A} f = \min_{\overline{A}} f$ (resp. $\max_{\partial A} f = \max_{\overline{A}} f$).*

The discrete p-capacity plays an important role in the first part of this paper.

Definition 2.9. Let $p > 0$ and let $A_i \subseteq V$ $(i = 0, 1, 2)$ with $A_0 \cup A_1 \subseteq A_2$. Define the p-capacity between A_0 and A_1 in A_2 by

$$\mathrm{cap}_p^G(A_0, A_1; A_2) = \inf\bigl\{\mathcal{E}_{p,A_2}^G(f) \bigm| f \in \mathbb{R}^V, f = 0 \text{ on } A_0 \text{ and } f = 1 \text{ on } A_1\bigr\}.$$

We write $\mathrm{cap}_p^G(A_0, A_1)$ for $\mathrm{cap}_p^G(A_0, A_1; V)$. The underlying graph G is omitted in these notations if no confusion can occur.

The following monotonicity of p-capacity is immediate from the definition.

Lemma 2.10. *Let $p > 0$ and let $A_i \subseteq V$ $(i = 0, 1, 2)$. If $A_i' \subseteq A_i$ $(i = 0, 1)$, then*

$$\mathrm{cap}_p^G(A_0', A_1'; A_2) \leq \mathrm{cap}_p^G(A_0, A_1; A_2)$$

Typical p-harmonic functions are given as *equilibrium potential of p-capacity:*

Lemma 2.11 ([HS97a, Theorems 3.5 and 3.11])**.** *Let $p > 1$. Let $A_0, A_1 \subseteq V$ and let A_2 be non-empty connected subset of V with $A_0 \cap A_1 = \emptyset$ and $A_0 \cup A_1 \subseteq A_2$. There exists a unique function (called* equilibrium potential*) $\varphi\colon A_2 \to [0, 1]$ such that $\varphi|_{A_i} \equiv i$ for $i = 0, 1$ and $\mathcal{E}_{p,A_2}^G(\varphi) = \mathrm{cap}_p^G(A_0, A_1; A_2)$. Furthermore, φ is p-harmonic in $A_2 \setminus (A_0 \cup A_1)$.*

On bounded degree graphs, the notions of modulus and capacity between sets are comparable as observed by He and Schramm [HS95, Theorem 8.1].

Lemma 2.12 (e.g.[Kig20, Proposition 4.8.4])**.** *Let $p > 0$. Then there exists $C \geq 1$ depending only on $p, \deg(G)$ such that the following statement is true: for any $A_i \subseteq V (i = 0, 1, 2)$ with $A_0 \cup A_1 \subseteq A_2$,*

$$C^{-1}\mathrm{cap}_p^G(A_0, A_1; A_2) \leq \mathrm{Mod}_p^G(A_0, A_1; A_2) \leq C\mathrm{cap}_p^G(A_0, A_1; A_2). \tag{2.4}$$

2.4 Volume growth conditions

We recall doubling properties and Ahlfors regularity on graphs and metric spaces.

Definition 2.13. A metric space (X, d) is said to be *metric doubling* if there exists $N_{\mathrm{D}} \in \mathbb{N}$ such that any ball $B_{\mathsf{d}}(x,r)$ can be covered by at most N_{D} balls with radii $r/2$. A Borel measure $\mathfrak{m}$ on X is said to be *volume doubling* (VD for short) with respect to d if there exists $C_{\mathrm{D}} \geq 1$ such that

$$0 < \mathfrak{m}(B_{\mathsf{d}}(x,2r)) \leq C_{\mathrm{D}}\, \mathfrak{m}(B_{\mathsf{d}}(x,r)) < \infty \quad \text{for all } x \in X,\ r > 0. \tag{VD}$$

A graph $G = (V, E)$ is *volume doubling* if VD holds with respect to the graph distance and the counting measure.

Definition 2.14. Let $d_{\mathrm{f}} > 0$. A metric space (X, d) is said to be d_{f}*-Ahlfors regular* (AR(d_{f}) for short) if there exist $C_{\mathrm{AR}} \geq 1$ and a Borel measure $\mathfrak{m}$ on X with

$$C_{\mathrm{AR}}^{-1}\, r^{d_{\mathrm{f}}} \leq \mathfrak{m}(B_d(x,r)) \leq C_{\mathrm{AR}}\, r^{d_{\mathrm{f}}} \qquad \text{for any } x \in X \text{ and } r \in (0, 2\operatorname{diam}(X, \mathsf{d})). \tag{AR(d_{f})}$$

We also say that $\mathfrak{m}$ is d_{f}*-Ahlfors regular* in such a case. The metric space (X, d) or a Borel measure $\mathfrak{m}$ is said to be *Ahlfors regular* if it satisfies AR(d_{f}) for some $d_{\mathrm{f}} > 0$. We shall say that a graph $G = (V, E)$ is d_{f}*-Ahlfors regular* if the condition above defining AR(d_{f}) holds with respect to the graph distance and the counting measure for all $x \in V$ and for all $r \in [1, \operatorname{diam}(V) + 1)$.

We recall a few elementary consequences of these definitions.

Remark 2.15. Let (X, d) be a metric space.

(1) If there exists a volume doubling measure $\mathfrak{m}$ on (X, d), then (X, d) is metric doubling whose doubling constant N_{D} depends only on the doubling constant C_{D} of $\mathfrak{m}$. [Hei, Chapter 13]

(2) If a Borel measure $\mathfrak{m}$ on X satisfies AR(d_{f}) for some $d_{\mathrm{f}} > 0$, then $\mathfrak{m}$ is volume doubling whose doubling constant C_{D} depends only on C_{AR} and $d_{\mathrm{f}} > 0$. Furthermore, AR(d_{f}) implies that the Hausdorff dimension of (X, d) is d_{f}.

We recall the following consequence of the volume doubling property.

Lemma 2.16. *Let (X, d) be a metric space and let $\mathfrak{m}$ be a Borel measure on X satisfying VD. Then there exists $\alpha > 0$ depending only on the doubling constant C_{D} such that*

$$\frac{\mathfrak{m}(B_{\mathsf{d}}(x,R))}{\mathfrak{m}(B_{\mathsf{d}}(y,r))} \leq C_{\mathrm{D}}^2 \left(\frac{\mathsf{d}(x,y) + R}{r} \right)^{\alpha} \quad \textit{for any } x, y \in X \textit{ and } 1 \leq r \leq R < \infty. \tag{VD(α)}$$

In particular,

$$\mathfrak{m}(B_{\mathsf{d}}(x,R)) \leq C_{\mathrm{D}} R^{\alpha} \quad \textit{for any } x \in X \textit{ and } 1 \leq R < \operatorname{diam}(X, \mathsf{d}). \tag{2.5}$$

Since increasing α does not affect the validity of VD(α), we assume that $\alpha \geq 1$ for much of this work.

3 Loewner-type lower bounds for p-modulus

Throughout this section, let $p \geq 1$ and let $G = (V, E)$ be a locally finite connected simple non-directed graph.

We introduce the following Loewner-type lower bounds on modulus between balls. The case with exponent $\zeta = 0$ was introduced by Bonk and Kleiner [BK05, Proposition 3.1]. This was extended by Bourdon and Kleiner [BK13, Proposition 2.9] to a discrete setting.

Definition 3.1. Let $\zeta \in \mathbb{R}$. A graph G satisfies *p-combinatorial ball Loewner condition with exponent ζ* ($\mathrm{BCL}_p(\zeta)$ for short) if there exists $A \geq 1$ such that the following hold: for any $\kappa > 0$ there exist $c_{\mathrm{BCL}}(\kappa) > 0$ and $L_{\mathrm{BCL}}(\kappa) > 0$ such that

$$\mathrm{Mod}_p^G(\{\theta \in \mathsf{Path}(B_1, B_2) \mid \operatorname{diam}\theta \leq L_{\mathrm{BCL}}(\kappa)R\}) \geq c_{\mathrm{BCL}}(\kappa)R^{\zeta} \tag{$\mathrm{BCL}_p(\zeta)$}$$

whenever $R \in [1, \operatorname{diam}(G)/A)$ and $B_i\,(i = 1, 2)$ are balls with radii R satisfying $\operatorname{dist}(B_1, B_2) \leq \kappa R$.

In this section, we discuss $\mathrm{BCL}_p(\zeta)$ and prove a key estimate (Theorem 3.2) in this paper. The setting of this section is given by the following condition:

$$\text{The underlying graph } G \text{ satisfies } \mathrm{BCL}_p(\zeta) \text{ and } 1 - p \leq \zeta < 1. \tag{$\mathrm{BCL}_p^{\mathrm{low}}(\zeta)$}$$

We are interested in the case where ζ is the 'largest' possible value. Since $\mathrm{BCL}_p^{\mathrm{low}}(1-p)$ is always true by (2.2), there is not much loss of generality in the assumption $\zeta \geq 1 - p$ but the inequality $\zeta < 1$ need not be true in general but holds in many 'low dimensional settings' such as the Sierpiński carpet.

Under $\mathrm{BCL}_p^{\mathrm{low}}(\zeta)$, we can show a generalized lower bound of p-modulus as in the next theorem, which is one of the main results in this section. It states that Loewner-type lower bounds on modulus between balls imply analogous lower bound on modulus between any pair of connected sets. This result plays important roles in the proofs of Poincare inequality in §4 and elliptic Harnack inequality in §5. The following theorem can be viewed as an extension of a result of Bonk and Kleiner from $\zeta = 0$ to more general exponent ζ [BK05, Proposition 3.1], [BK13, Proposition 2.9].

Theorem 3.2. *Assume that G is bounded degree graph that satisfies p-combinatorial ball Loewner condition $\mathrm{BCL}_p^{\mathrm{low}}(\zeta)$ with exponent $\zeta \in [1-p, 1)$, where $p \geq 1$. Let $\kappa_0 > 0$. Then there exist $c, L > 0$ depending only on the constants associated to the assumptions such that the following holds: If $F_i\,(i = 1, 2)$ are disjoint connected subsets of V that satisfy*

$$\frac{\operatorname{dist}(F_1, F_2)}{\operatorname{diam} F_1 \wedge \operatorname{diam} F_2} \leq \kappa_0,$$

then

$$\mathrm{Mod}_p^G(\{\theta \in \mathsf{Path}(F_1, F_2) \mid \operatorname{diam}\theta \leq LR_0\}) \geq cR_0^{\zeta}, \tag{3.1}$$

where $R_0 \coloneqq 2\operatorname{dist}(F_1, F_2) \wedge \frac{1}{2}\operatorname{diam} F_1 \wedge \frac{1}{2}\operatorname{diam} F_2$.

Similar to [BK05, BK13], the idea behind its proof is to show the existence of a shortcut with respect to an arbitrary function $\rho \in \ell^+(V)$ and use Lemma 2.4. The following lemma is a key ingredient, which is a discrete analogue of [BK05, Lemma 3.7] (see also [BK13, Lemma 2.10]). We omit its proof because it is essentially the same as [BK05, Lemma 3.7].

Lemma 3.3. *Suppose that $G = (V, E)$ satisfies $\mathrm{BCL}_p(\zeta)$. For any $\lambda \in (0, 1/8)$, let $L_\lambda \coloneqq L_{\mathrm{BCL}}\left(\frac{9}{2\lambda}\right) + \frac{7}{8}$. Let (B, F_1, F_2) be a triple such that $B = B(x, R)$ for some $x \in V$ and $R \geq 16$ and F_i $(i = 1, 2)$ are connected subset of V. If the triple (B, F_1, F_2) satisfies*

$$F_i \cap \frac{1}{4}B \neq \emptyset \quad \text{and} \quad F_i \setminus B \neq \emptyset \quad (i = 1, 2), \tag{3.2}$$

then for any $\rho \in \ell^+(V)$ there exist $x_i \in F_i$ $(i = 1, 2)$ satisfying the following properties:

(i) *For each $i = 1, 2$, $x_i \in \overline{B}(x, 3R/4)$ and $d(x, x_1) \wedge d(x, x_2) \leq 3R/8$. Furthermore, $B_i \coloneqq B(x_i, \lambda R)$ satisfies $\frac{1}{8\lambda}B_i \subseteq \frac{7}{8}B$ and $B_1 \cap B_2 = \emptyset$.*

(ii) *There exists a constant $C_{\mathrm{shr}} > 0$ (we can take $C_{\mathrm{shr}} = 128$) such that $\|\rho\|_{p,B_i}^p \leq C_{\mathrm{shr}}(\lambda \vee R^{-1}) \|\rho\|_{p,B}^p$ for each $i = 1, 2$.*

(iii) *There exists $\theta \in \mathsf{Path}\left(\frac{1}{4}B_1, \frac{1}{4}B_2\right)$ such that $\theta \subseteq L_\lambda B$, $\operatorname{diam}\theta \leq L_{\mathrm{BCL}}\left(\frac{9}{2\lambda}\right)R$ and*

$$L_\rho(\theta) \leq C_{p,\lambda}(\lambda R)^{-\zeta/p} \|\rho\|_{p,L_\lambda B},$$

where $C_{p,\lambda} > 0$ is a constant depending only on p, ζ, λ and $c_{\mathrm{BCL}}\left(\frac{9}{2\lambda}\right)$.

(iv) *$F_i \cap \frac{1}{4}B_i$, $\theta \cap \frac{1}{4}B_i$, $F_i \setminus B_i$ and $\theta \setminus B_i$ $(i = 1, 2)$ are non-empty.*

Let us sketch to the proof of main result (Theorem 3.2) using Lemma 3.3.

Sketch of the proof of Theorem 3.2. Since the proof is essentially same as the proofs of [BK05, Proposition 3.1] and [BK13, Proposition 2.9], we only sketch the argument.

Let $\rho \in \ell^+(V)$. By Lemma 2.4, it suffices to show the existence of a path θ with $\operatorname{diam}(\theta) \leq LR_0$ and $L_\rho(\theta) \lesssim R_0^{-\zeta/p} \|\rho\|_p$.

First we choose balls B_1 and B_2 centered at F_1 and F_2 respectively with radii λR_0. A use of Lemma 2.4 and the ball combinatorial Loewner condition implies that there exists L_ρ-shortcuts between balls B_1 and B_2, say of radius $\lambda R_0/4$ where $\lambda < 1/8$ centered at the two connected sets (see Lemma 3.3). The two gaps at scale λR_0 are now inductively filled by paths between suitably chosen balls creating four gaps at scale $\lambda^2 R_0$. Note that the Lemma 3.3 can be inductively applied to 2^k gaps at scale $\lambda^k R_0$ as long as $\lambda^k R_0 > 16$ since the condition (3.2) is guaranteed by Lemma 3.3(iv). The balls are chosen so that the ρ-mass decays linearly with the radius (cf. [BK05, Lemma 3.5] and Lemma 3.3(ii)). Continuing inductively, we obtain L_ρ-shortcuts that the total length of gaps at scale $\lambda^k R$ with $\lambda^k R_0 \gtrsim 1$ is of the order $(C\lambda)^{k(1-\zeta)/p} R^{-\zeta/p}$, where $C\lambda < 1$. If $\lambda^k R_0 \leq 16$, we use (2.2) in Lemma 2.4 to fill the gaps at the smallest scale. The ρ-length of this shortcut can therefore be bounded by a geometric series $\sum_{k=0}^{\infty} (C\lambda)^{k(1-\zeta)/p} R_0^{-\zeta/p}$. This geometric series converges and provides the desired shortcut if $\zeta < 1$. □

We also frequently use the following consequence of Theorem 3.2.

Corollary 3.4. *Assume that G is bounded degree graph that satisfies p-combinatorial ball Loewner condition $\mathrm{BCL}_p^{\mathrm{low}}(\zeta)$ with exponent $\zeta \in [1-p, 1)$, where $p \geq 1$. There exist $c > 0$ and $L \geq 1$ depending only on the constants associated with the assumptions such that if $F_i\,(i = 1, 2)$ are connected subsets of V satisfying $\#F_i \geq 2$, $F_i \cap B \neq \emptyset$ and $F_i \setminus 4B \neq \emptyset$ for some ball B with radius $R > 0$, then*

$$\mathrm{Mod}_p^G(F_1, F_2; 4LB) \geq c(R \vee 1)^{\zeta}. \tag{3.3}$$

Proof. We first consider the case $R \geq 2$. Notice that $V \setminus 4B \neq \emptyset$. Since F_i is connected, we can find a connected subset $\widetilde{F}_i$ of F_i satisfying the following conditions (i)-(iii):

(i) $\widetilde{F}_1 \subseteq F_1 \cap \bigl(\overline{2B} \setminus B\bigr)$ and $\widetilde{F}_2 \subseteq F_2 \cap \bigl(\overline{4B} \setminus 3B\bigr)$.

(ii) $\widetilde{F}_1 \cap \overline{B} \neq \emptyset$ and $\widetilde{F}_2 \cap \overline{3B} \neq \emptyset$.

(iii) $\widetilde{F}_1 \setminus 2B \neq \emptyset$ and $\widetilde{F}_2 \setminus 4B \neq \emptyset$.

Then we immediately see that $3R \geq \operatorname{diam} \widetilde{F}_1 \geq \operatorname{diam} \widetilde{F}_2 = \lceil 4R \rceil - \lceil 3R \rceil \geq \frac{1}{2}R$ and

$$8R \geq \operatorname{dist}\bigl(\widetilde{F}_1, \widetilde{F}_2\bigr) \geq \lceil 3R \rceil - \lceil 2R \rceil \geq \frac{1}{2}R.$$

Hence, by applying Theorem 3.2 for $\widetilde{F}_i$, there exist $c, L > 0$ depending only on the constants associated with the assumptions such that

$$\mathrm{Mod}_p\Bigl(\bigl\{\theta \in \mathsf{Path}\bigl(\widetilde{F}_1, \widetilde{F}_2\bigr) \bigm| \operatorname{diam}\theta \leq LR\bigr\}\Bigr) \geq cR^{\zeta}.$$

By Lemma 2.3(ii),

$$\mathrm{Mod}_p^G(\{\theta \in \mathsf{Path}(F_1, F_2) \mid \theta \subseteq (L+1)B\}) \geq \mathrm{Mod}_p^G\Bigl(\bigl\{\theta \in \mathsf{Path}\bigl(\widetilde{F}_1, \widetilde{F}_2\bigr) \bigm| \operatorname{diam}\theta \leq LR\bigr\}\Bigr),$$

which implies our assertion in this case.

Next we consider the case $R \leq 2$. Let $L > 0$ be the same as in the previous paragraph. Then, by (2.2) in Lemma 2.4, we have

$$\begin{aligned}
&\mathrm{Mod}_p^G(\{\theta \in \mathsf{Path}(F_1, F_2) \mid \theta \subseteq (L+4)B\}) \\
&\geq \mathrm{Mod}_p^G(\{\theta \in \mathsf{Path}(F_1, F_2) \mid \theta \text{ is a shortest path}\}) \\
&\geq 4^{1-p} = 4^{1-p}(R \vee 1)^{-\zeta} \cdot (R \vee 1)^{\zeta} \geq 4^{1-p}\bigl(2^{-1} \wedge 1\bigr)(R \vee 1)^{\zeta},
\end{aligned}$$

where we used $(R \vee 1)^{-\zeta} \geq (R \vee 1)^{-1} \wedge 1^{p-1}$ and $R \leq 2$ in the last inequality. $\square$

4 Discrete (p, p)-Poincaré inequality

Throughout this section, let $p \geq 1$ and let $G = (V, E)$ be a locally finite connected simple non-directed graph.

The goal of this section is to show that the 'low-dimensional' p-ball combinatorial Loewner type property $\mathrm{BCL}_p^{\mathrm{low}}(\zeta)$ implies a Poincaré inequality. We shall give the definition of (weak) (p, p)-Poincaré inequality in our setting.

Definition 4.1. Let $\beta > 0$. A graph G satisfies (p, p)-*Poincaré inequality of order* β ($\mathrm{PI}_p(\beta)$ for short) if there exist $C_{\mathrm{PI}}, A_{\mathrm{PI}} \geq 1$ such that for any $x \in V$, $R \geq 1$ and $f \in \mathbb{R}^V$,

$$\sum_{y \in B(x,R)} \left| f(y) - f_{B(x,R)} \right|^p \leq C_{\mathrm{PI}} R^{\beta} \mathcal{E}^G_{p, B(x, A_{\mathrm{PI}} R)}(f). \tag{$\mathrm{PI}_p(\beta)$}$$

The main result in this section (Theorem 4.2) shows that the (p, p)-*Poincaré inequality* follows from the the combinatorial ball Loewner-type property $\mathrm{BCL}_p^{\mathrm{low}}(\zeta)$ and VD. This result and its proof are inspired by a similar theorem of Heinonen and Koskela [HK98, Theorem 5.12]. Although the result in [HK98] corresponds to the case $\zeta = 0$ the proof there works when $\zeta < 1$.

Theorem 4.2. *Let $G = (V, E)$ be a graph satisfying $\mathrm{VD}(\alpha)$ and $\mathrm{BCL}_p^{\mathrm{low}}(\zeta)$, where $\alpha \geq 1$ and $\zeta \in [1 - p, 1)$. Then G satisfies $\mathrm{PI}_p(\beta)$, where $\beta = \alpha - \zeta$, $A_{\mathrm{PI}} = 2$ and C_{PI} depends only on the constants associated with the assumptions.*

The proof of Theorem 4.2 is done in two steps. In the first step, we introduce a two-point estimate that is a sufficient condition for the Poincaré inequality (see Definition 4.3 and Lemma 4.5). In the second step, we show that the combinatorial ball Loewner-type property $\mathrm{BCL}_p^{\mathrm{low}}(\zeta)$ implies the two-point estimate (Lemma 4.6).

The following definition gives a discrete generalization of pointwise estimates (see [HK00, (15)] or [HK98, (5.16)] for example).

Definition 4.3. Let $\beta > 0$. The graph G satisfies the p-*two-point estimate of order* β ($\mathrm{TP}_p(\beta)$ for short) if there exists $C_{\mathrm{TP}} > 0$ such that for any $z \in V$, $R \geq 1$, $f \in \mathbb{R}^V$ and $x, y \in B(z, C_{\mathrm{TP}}^{-1} R)$,

$$|f(x) - f(y)|^p \leq C_{\mathrm{TP}} R^{\beta} \left(\max_{r \in (0,R)} \frac{\mathcal{E}^G_{p, B(x,r)}(f)}{\# B(x,r)} + \max_{r \in (0,R)} \frac{\mathcal{E}^G_{p, B(y,r)}(f)}{\# B(x,r)} \right). \tag{$\mathrm{TP}_p(\beta)$}$$

It is easy to see that $\mathrm{VD}(\alpha)$, where $\alpha \geq 1$, implies $\mathrm{TP}_p(\alpha + p - 1)$.

A well-known telescoping sum argument show that Poincaré inequality implies the two point estimate. This follows from a straighforward modification of the proof of [HK98, Lemma 5.15] or a discrete version of that argument in the special case $p = 2$ in [Mur20, Lemma 2.4]. We omit its proof as we will not use the lemma below.

Lemma 4.4. *Let $G = (V, E)$ be a graph satisfying VD and $\mathrm{PI}_p(\beta)$ for some $\beta > 0$. Then G satisfies $\mathrm{TP}_p(\beta)$.*

The following lemma is a converse of the previous lemma, which can be shown by following [HK98, Lemma 5.15] with minor modifications. Let us recall the notion of *median*. For $f \in \mathbb{R}^V$ and $A \subseteq V$, a median of f on A is a number $a \in \mathbb{R}$ such that

$$\#\{w \in A \mid f(z) \geq a\} \wedge \#\{w \in A \mid f(z) \leq a\} \geq \frac{1}{2}\#A.$$

We write $\mathsf{med}(f, A)$ to denote the set of medians of f on A. Note that $\mathsf{med}(f, A) \neq \emptyset$.

Lemma 4.5. *Let $G = (V, E)$ be a graph satisfying VD and $\mathrm{TP}_p(\beta)$ for some $\beta > 0$. Then there exist $C > 0$ and $A > 0$ depending only on $p, C_{\mathrm{D}}, \deg(G)$, C_{TP} such that*

$$\sum_{B(x,R)} |f - a|^p \leq C R^{\beta} \mathcal{E}^G_{p,B(x,AR)}(f), \tag{4.1}$$

for any $x \in V$, $R \geq 1$, $f \in \mathbb{R}^V$, $a \in \mathsf{med}\big(f, B(x, R)\big)$. In particular, G satisfies $\mathrm{PI}_p(\beta)$ by [BB, Lemma 4.17].

Finally we prove $\mathrm{PI}_p(\alpha - \zeta)$ for a graph G satisfying $\mathrm{BCL}_p^{\mathrm{low}}(\zeta)$ and $\mathrm{VD}(\alpha)$ with exponent $\alpha \geq 1$. By virtue of Lemma 4.5, it is enough to show the following lemma.

Lemma 4.6. *Let $G = (V, E)$ be a graph satisfying $\mathrm{VD}(\alpha)$ and $\mathrm{BCL}_p^{\mathrm{low}}(\zeta)$. Then G satisfies $\mathrm{TP}_p(\alpha - \zeta)$ and the associated constant C_{TP} depends only on constants involved in the assumptions.*

Sketch of the proof. The proof is an adaptation of [HK98, Proof of Lemma 5.17] which we briefly recall. This is a proof by contradiction. Suppose to the contrary that there is a function $|f(x) - f(y)| = 1$ such that

$$\max_{r \in (0, CR)} \frac{\mathcal{E}^G_{p,B(x,r)}(f)}{\#B(x, r)} + \max_{r \in (0, CR)} \frac{\mathcal{E}^G_{p,B(y,r)}(f)}{\#B(y, r)} \leq \epsilon R^{-\beta}, \tag{4.2}$$

where $R = d(x, y)$, $C > 2$ and $\epsilon > 0$ is small enough. Set $\rho(v) \coloneqq \max_{e \in E; v \in e} |\nabla f|(e)$, $v \in V$, and $\beta = \alpha - \zeta > 0$.

Pick a geodesic path γ from x to y. By Lemma 2.4, (4.2) and Theorem 3.2, for $C_2 > 1$ and $1 < r < C_2 r < R/2$, there exists a curve θ_r joining $B(z, r) \cap \gamma$ to $B(z, C_2 r)^c \cap \gamma$ where $z \in \{x, y\}$ such that

$$L_\rho(\theta_r) \lesssim \epsilon^{1/p} (r/R)^{\beta/p}.$$

By choosing r's along a geometric sequence of scales $K^{-j}R, j \in \mathbb{N}$ and joining shortcuts $\theta_{K^{-j}R}$ and $\theta_{K^{-j-1}R}$ again by using Lemma 2.4, (4.2) and Theorem 3.2 yields a L_ρ-shortcut θ connecting x and y such that

$$L_\rho(\theta) \lesssim \epsilon^{1/p} \sum_{j=0}^{\infty} K^{-j\beta/p} \leq C_3 \epsilon^{1/p}$$

where C_3 only depends on the constants associated with the assumptions. By the triangle inequality $L_\rho(\theta) \geq |f(x) - f(y)| = 1$ and hence we obtain the desired contradiction if $\epsilon < C_3^{-p}$. □

Proof of Theorem 4.2. Combining Lemmas 4.5 and 4.6, we obtain Theorem 4.2. □

5 Discrete elliptic Harnack inequality

This section is devoted to Harnack type inequalities for discrete p-harmonic functions. Such estimates are crucial to establish that the Sobolev space we construct has a dense set of continuous functions.

Throughout this section, let $p \in (1, \infty)$ and let $G = (V, E)$ be a locally finite connected simple non-directed graph.

5.1 EHI for discrete p-harmonic functions

The Poincaré inequality introduced in Definition 4.1 implies a lower bound on capacity across annulus. Let us introduce a matching capacity upper bound which serves to identify the exponent β introduced in Definition 4.1 as the best possible one.

Definition 5.1. Let $\beta > 0$. A graph G satisfies $\mathrm{cap}_{p,\leq}(\beta)$ if there exist $C_{\mathrm{cap}} > 0$ and $A_{\mathrm{cap}} \geq 1$ such that for any $x \in V$ and $R \in [1, \mathrm{diam}(G)/A_{\mathrm{cap}})$,

$$\mathrm{cap}_p^G\bigl(B(x,R), B(x,2R)^c\bigr) \leq C_{\mathrm{cap}} \frac{\#B(x,R)}{R^\beta}. \qquad (\mathrm{cap}_{p,\leq}(\beta))$$

The following generalization of $\mathrm{cap}_{p,\leq}(\beta)$ is well-known and done by a standard covering argument using the metric doubling property.

Lemma 5.2. *Let $d_{\mathrm{f}} \geq 1, \beta > 0$ and let $G = (V, E)$ satisfy $\mathrm{AR}(d_{\mathrm{f}})$ and $\mathrm{cap}_{p,\leq}(\beta)$. For any $\delta \in (0, 1)$ there exists $C_{\mathrm{cap}}(\delta) > 0$ depending only on δ and the constants associated with the assumptions such that for any $x \in V$ and $R \geq \delta^{-1}$,*

$$\mathrm{cap}_p^G\bigl(B(x,\delta R), B(x,R)^c\bigr) \leq C_{\mathrm{cap}}(\delta) \frac{\#B(x,\delta R)}{R^\beta}.$$

To prove Harnack type inequality, the log-Caccioppoli inequality (e.g. [HS97b, 2.12 Corollary]) is a standard technique. The proof for the case $p = 2$ in [KZ92, (7.5) Lemma] extends easily to the general case $p \in (1, \infty)$

Lemma 5.3 (Log-Caccioppoli inequality)**.** *Let $p \in (1, \infty)$. Let $A \subseteq V$ and $\varphi \colon V \to [0, 1]$ with* $\mathrm{supp}[\varphi] \subseteq A$*. If $h \colon V \to (0, \infty)$ is p-superharmonic in A, then for some constant $C_p > 0$ depending only on p,*

$$\sum_{\{x,y\} \in E(\overline{A})} (\varphi(x)^p \wedge \varphi(y)^p) \lvert \log h(x) - \log h(y) \rvert^p \leq C_p \, \mathcal{E}_p^G(\varphi). \qquad (5.1)$$

The main result of this section is the following elliptic Harnack inequality. A similar proof in the classical setting can be found in [Hol03, Theorem 4.3].

Theorem 5.4. *Let $p \in (1, \infty)$, $d_{\mathrm{f}} \geq 1$ and $\beta > 0$. Assume that G satisfies $\mathrm{AR}(d_{\mathrm{f}})$, $\mathrm{BCL}_p^{\mathrm{low}}(d_{\mathrm{f}} - \beta)$ and $\mathrm{cap}_{p,\leq}(\beta)$. Then there exist $\delta_{\mathrm{H}} \in (0, 1)$ and $C_{\mathrm{H}} \geq 1$ depending only on* *the constants associated with the assumptions such that, for any $x \in V$ and $R \geq 1$ with $B(x,R) \neq V$, if $h\colon V \to [0,\infty)$ is p-harmonic in $B(x,R)$, then*

$$\max_{B(x,\delta_{\mathrm{H}}R)} h \leq C_{\mathrm{H}} \min_{B(x,\delta_{\mathrm{H}}R)} h. \tag{5.2}$$

Proof. Fix $\delta_{\mathrm{H}} \in \bigl(0,(4L)^{-1}\bigr)$, where L is the constant appeared in Corollary 3.4. By Lemma 2.3, we can assume that $L \geq 2$ without loss of generality. Let $\varepsilon > 0$ and set $h_\varepsilon := h + \varepsilon$. Note that h_ε is also p-harmonic on $B := B(x,R)$. Define

$$m := \min_{B(x,\delta_{\mathrm{H}}R)} h_\varepsilon \quad \text{and} \quad M := \max_{B(x,\delta_{\mathrm{H}}R)} h_\varepsilon.$$

If $R \leq 4L$, then $B(x,\delta_{\mathrm{H}}R) = \{x\}$ and thus $m = M$. Hence it is enough to consider the case $R \geq 4L$. In this case, we always have $R - \delta_{\mathrm{H}}R > 4L - 1 > 2$, in particular $B(x,R) \setminus B(x,\delta_{\mathrm{H}}R) \neq \emptyset$. Using the maximum/minimum principles (Lemma 2.8), we can find paths $\theta_{\min}, \theta_{\max}$ in G satisfying the following conditions (i) and (ii).

(i) $\theta_{\min} \subseteq \{h_\varepsilon \leq m\}$ and $\theta_{\max} \subseteq \{h_\varepsilon \geq M\}$;

(ii) $\theta_{\min}, \theta_{\max} \in \mathsf{Path}\bigl(\partial_i B(x,\delta_{\mathrm{H}}R), \partial_i B(x,R); B(x,R)\bigr)$.

Set $\delta := 4\delta_{\mathrm{H}}L \in (0,1)$. Since $B(x,4\delta_{\mathrm{H}}R) \subseteq B(x,\frac{1}{2}B)$ by $L \geq 2$, it follows from Corollary 3.4 that there exists $c > 0$ depending only on the constants associated with the assumptions such that

$$\mathrm{Mod}_p^G(\theta_{\min},\theta_{\max};\delta B) \geq c\, R^{d_{\mathrm{f}}-\beta}. \tag{5.3}$$

In order to show (5.2), it suffices to consider the case $m < M$. Define $h'_\varepsilon = \frac{1}{\log M - \log m}(\log h_\varepsilon - \log m)$ and $h^*_\varepsilon = (h'_\varepsilon \vee 0) \wedge 1$. Then we easily see that $\widetilde{h^*_\varepsilon} \in \mathsf{Adm}(\theta_{\min},\theta_{\max})$, where $\widetilde{h^*_\varepsilon}\colon V \to [0,\infty)$ is defined as

$$\widetilde{h^*_\varepsilon}(x) := \max_{y \in V; \{x,y\} \in E} |h^*_\varepsilon(x) - h^*_\varepsilon(y)| \quad \text{for } x \in V.$$

Noting that $m \geq \varepsilon > 0$, we have

$$\mathrm{Mod}_p^G(\theta_{\min},\theta_{\max};\delta B) \leq C\mathcal{E}^G_{p,\delta B}\bigl(\widetilde{h^*_\varepsilon}\bigr) \leq C\deg(G)\left(\log\frac{M}{m}\right)^{-p}\mathcal{E}^G_{p,\delta B}(\log h_\varepsilon), \tag{5.4}$$

where $C \geq 1$ is the constant in Lemma 2.12.

Let φ be the equilibrium potential of $\mathrm{cap}_p^G(\delta B, B^c)$ such that $\varphi\big|_{\delta B} \equiv 1$ and $\varphi\big|_{B^c} \equiv 0$. Since h_ε is positive and p-harmonic function in B, by applying the log-Caccioppoli inequality (Lemma 5.3) for the tuple (h,φ), we obtain

$$\mathcal{E}^G_{p,\delta B}(\log h_\varepsilon) \leq C_p \mathrm{cap}_p^G(\delta B, B^c). \tag{5.5}$$

From (5.3), (5.4), (5.5), $\mathrm{cap}_{p,\leq}(\beta)$, Lemma 5.2 and (2.5), we obtain

$$cR^{d_{\mathrm{f}}-\beta} \leq C_p C_{\mathrm{cap}}(\delta) \cdot C_{\mathrm{AR}}\delta^{d_{\mathrm{f}}}\deg(G)\left(\log\frac{M}{m}\right)^{-p} R^{d_{\mathrm{f}}-\beta},$$

which implies

$$\log \frac{M}{m} = \log \frac{\max_{\delta_{\mathrm{H}}B} h + \varepsilon}{\min_{\delta_{\mathrm{H}}B} h + \varepsilon} \le \Big(c^{-1} C_p C_{\mathrm{cap}}(4L\delta_{\mathrm{H}}) \cdot C_{\mathrm{AR}}(4L\delta_{\mathrm{H}})^{d_{\mathrm{f}}} \deg(G) \Big)^{1/p} := \log C_{\mathrm{H}}.$$

Hence,

$$\max_{\delta_{\mathrm{H}}B} h + \varepsilon \le C_{\mathrm{H}} \left(\min_{\delta_{\mathrm{H}}B} h + \varepsilon \right).$$

Since $\varepsilon > 0$ is arbitrary, (5.2) holds. □

A standard argument using Moser's oscillation lemma immediately yields the following interior Hölder regularity of harmonic functions (see [Sal02, §2.3.2] or [Bar, Proposition 1.45]). Recall the definition of oscillation $\operatorname{osc}_B[h]$ below from Notation (7).

Corollary 5.5. *Let $p \in (1, \infty)$, $d_{\mathrm{f}} \ge 1$ and $\beta > 0$. Assume that G satisfies* $\mathrm{AR}(d_{\mathrm{f}})$*,* $\mathrm{BCL}_p^{\mathrm{low}}(d_{\mathrm{f}} - \beta)$ *and* $\mathrm{cap}_{p,\le}(\beta)$*. For any $\lambda \in (0,1)$ there exist $C_{\mathrm{Höl}}, \theta_{\mathrm{Höl}} > 0$ depending only on the constants associated with the assumptions such that for any non-negative function $h \in \mathbb{R}^V$ which is p-harmonic in a ball B with radius $R \ge 1$,*

$$|h(x) - h(y)| \le C_{\mathrm{Höl}} \left(\frac{d_G(x,y)}{R} \right)^{\theta_{\mathrm{Höl}}} \operatorname*{osc}_B [h], \quad \text{for all } x, y \in \lambda B. \tag{5.6}$$

5.2 Hölder continuous cutoff functions with controlled energy

In this subsection, we construct *globally* Hölder continuous cutoff functions with controlled energy. Although energy minimizers for capacity are p-harmonic, the local Hölder regularity given by Corollary 5.5 is not sufficient to conclude the desired global Hölder regularity asserted in Theorem 5.6. This requires an additional Harnack-type estimate near boundaries.

The following theorem asserts the existence of Hölder continuous cutoff functions with controlled energy and is the main result in this subsection. This will in turn be used to show that our Sobolev spaces have a dense set of continuous functions.

Theorem 5.6. *Let $p \in (1, \infty)$, $d_{\mathrm{f}} \ge 1$, $\beta > 0$ and $K > 1$. Assume that G satisfies* $\mathrm{AR}(d_{\mathrm{f}})$*,* $\mathrm{BCL}_p^{\mathrm{low}}(d_{\mathrm{f}} - \beta)$ *and* $\mathrm{cap}_{p,\le}(\beta)$*. Then there exist $\theta_*, C_* > 0$ depending only on the constants associated to the assumptions such that the following hold: for any $z \in V$ and $R \ge 1$ with $B(z, KR) \ne V$, there exists a function $\varphi_{z,R} \colon V \to [0,1]$ satisfies*

$$\varphi_{z,R}\big|_{B(z,R)} \equiv 1, \quad \operatorname{supp}\big[\varphi_{z,R}\big] \subseteq B(z, KR), \tag{5.7}$$

$$\mathcal{E}_p^G(\varphi_{z,R}) \le C_* R^{d_{\mathrm{f}} - \beta}, \tag{5.8}$$

and

$$|\varphi_{z,R}(x) - \varphi_{z,R}(y)| \le C_* \left(\frac{d_G(x,y)}{R} \right)^{\theta_*} \quad \text{for any } x, y \in V. \tag{5.9}$$

Proof. Let $\delta_{\mathrm{H}} \in (0,1)$ be the constant in Theorem 5.4. Then we let

$$\delta_{*} \coloneqq \frac{K-1}{4\delta_{\mathrm{H}} + \delta_{\mathrm{H}}^{-1} + 1} \wedge \frac{K-1}{1 + 6\delta_{\mathrm{H}}^{-1}} \wedge \frac{\delta_{\mathrm{H}}^{2}}{10} > 0,$$

fix $\varepsilon \in [10^{-1}\delta_{*}, \delta_{*})$, and set $R_{*} \coloneqq \varepsilon^{-1}$. The case $1 \le R \le R_{*}$ follows by observing that the function

$$\varphi_{z,R}(x) \coloneqq \left(\frac{\lceil KR \rceil - d_{G}(z,x)}{\lceil KR \rceil - \lfloor R \rfloor}\right)^{+} \wedge 1.$$

satisfies the desired properties.

Hereafter, we consider the case $R \ge R_{*}$. Define

$$D \coloneqq B(z, KR) \setminus \left(\bigcup_{w \in \partial_{i} B(z,KR)} B\bigl(w, 2\varepsilon\delta_{\mathrm{H}}^{-1}R\bigr) \right),$$

and let $\varphi = \varphi_{z,R}$ be the equilibrium potential with respect to $\mathrm{cap}_{p}^{G}\bigl(B(z,R), D^{c}\bigr)$ satisfying $\varphi\big|_{B(z,R)} \equiv 1$ and $\mathrm{supp}[\varphi] \subseteq D$, which exists by Proposition 2.11. Note that $B(z,KR) \neq V$ implies $\partial_{i} B(z,KR) \neq \emptyset$. For any $w \in \partial_{i} B(z,KR)$ and $y \in B(w, 2\varepsilon\delta_{\mathrm{H}}^{-1}R)$,

$$\begin{aligned} d_{G}(z,y) \ge d_{G}(z,w) - d_{G}(w,y) &> \lfloor KR \rfloor - 2\varepsilon\delta_{\mathrm{H}}^{-1}R \\ &\ge (K - R^{-1} - 2\varepsilon\delta_{\mathrm{H}}^{-1})R \ge (K - \varepsilon - 2\varepsilon\delta_{\mathrm{H}}^{-1})R, \end{aligned}$$

which implies $B(z, K'R) \subseteq D$, where $K' \coloneqq K'(\varepsilon, \delta_{\mathrm{H}}, K) \coloneqq K - \varepsilon - 2\varepsilon\delta_{\mathrm{H}}^{-1} > 1$. Here we used $\varepsilon \le (K-1)/(1 + 6\delta_{\mathrm{H}}^{-1}) < (K-1)/(1 + 2\delta_{\mathrm{H}}^{-1})$ to ensure that $K' > 1$. By Lemma 2.10, $\mathrm{cap}_{p,\le}(\beta)$, $\mathrm{AR}(d_{\mathrm{f}})$ and Lemma 5.2,

$$\mathcal{E}_{p}^{G}(\varphi) = \mathrm{cap}_{p}^{G}\bigl(B(z,R), D^{c}\bigr) \le \mathrm{cap}_{p}^{G}\bigl(B(z,R), B(z,K'R)^{c}\bigr) \le C' R^{d_{\mathrm{f}} - \beta},$$

where $C' > 0$ depends only on the constants associated to the assumptions.

The rest is proving (5.9). It suffices to show the following Hölder regularity on each balls with radii εR: there exist $C, \theta > 0$ depending only on the constants associated with the assumptions such that

$$|\varphi(x) - \varphi(y)| \le C \left(\frac{d_{G}(x,y)}{R}\right)^{\theta} \quad \text{for all } z' \in D \text{ and } x, y \in B(z', \varepsilon R). \tag{5.10}$$

Fix $z' \in D$ and set $B_{*} \coloneqq B(z', 2\varepsilon R)$. We consider the following three cases.

Case 1: $\delta_{\mathrm{H}}^{-1} B_{*} \subseteq D \setminus B(z,R)$. Note that $\mathrm{osc}_{V}[\varphi] = 1$ and that φ is p-harmonic in $\delta_{\mathrm{H}}^{-1} B_{*}$. The estimate (5.10) follows from Corollary 5.5.

Case 2: $\delta_{\mathrm{H}}^{-1} B_{*} \cap B(z,R) \neq \emptyset$. Since $\mathrm{diam}\bigl(\delta_{\mathrm{H}}^{-1} B_{*}\bigr) \le 4\varepsilon\delta_{\mathrm{H}}^{-1} < K' - 1$ by $\varepsilon < (K-1)/(1 + 6\delta_{\mathrm{H}}^{-1})$, we have from $\delta_{\mathrm{H}}^{-1} B_{*} \cap B(z,R) \neq \emptyset$ that $\delta_{\mathrm{H}}^{-1} B_{*} \subseteq B(z, K'R) \subseteq D$. If $B_{*} \subseteq B(z,R)$, then $\max_{x,y \in B_{*}} |\varphi(x) - \varphi(y)| = |1 - 1| = 0$ and thus (5.10) is evident. In the rest of this part, we suppose $B(z,R) \setminus B_{*} \neq \emptyset$. Define

$$m_{*} \coloneqq \min_{B_{*}} \varphi \quad \text{and} \quad M_{*} \coloneqq \max_{B_{*}} \varphi.$$

Clearly, $0 \le m_* \le M_* \le 1$. By $B(z, KR) \neq V$, we note that $\partial_i \delta_{\mathrm{H}}^{-1} B_* \neq \emptyset$. Since φ is p-superharmonic in D, by the minimum principle (Lemma 2.8), there exists a path $\gamma_{\min}$ in G satisfying

$$\gamma_{\min} \in \mathsf{Path}(\partial_i B_*, \partial_i \delta_{\mathrm{H}}^{-1} B_*; \delta_{\mathrm{H}}^{-1} B_*) \quad \text{and} \quad \gamma_{\min} \subseteq \{\varphi \le m_*\}.$$

Since

$$\operatorname{diam} B_* + \operatorname{rad}\bigl(\delta_{\mathrm{H}}^{-1} B_*\bigr) \le \bigl(4 + \delta_{\mathrm{H}}^{-1}\bigr)\varepsilon R < \frac{\delta_{\mathrm{H}}}{2} \cdot R < R,$$

where we used $\varepsilon < \delta_{\mathrm{H}}^2/10 < \delta_{\mathrm{H}}^2/(2 + 8\delta_{\mathrm{H}})$ to ensure $(4 + \delta_{\mathrm{H}}^{-1})\varepsilon < 2^{-1}\delta_{\mathrm{H}}$, we obtain $z \notin \delta_{\mathrm{H}}^{-1} B_*$. This together with $\varphi\big|_{B(z,R)} \equiv \max_V \varphi = 1$ implies that there exists a path $\gamma_{\max}$ in G such that

$$\gamma_{\max} \in \mathsf{Path}(\partial_i B_*, \partial_i \delta_{\mathrm{H}}^{-1} B_*; \delta_{\mathrm{H}}^{-1} B_*) \quad \text{and} \quad \gamma_{\max} \subseteq \{\varphi \ge M_*\},$$

where we used the maximum principle (Lemma 2.8) on $D \setminus B(z, R)$ if necessary. Indeed, for any $x_0 \in \partial_i B(z, R) \cap \delta_{\mathrm{H}}^{-1} B_*$, there exists a path $\gamma_0 \in \mathsf{Path}(\{x_0\}, \partial_i \delta_{\mathrm{H}}^{-1} B_*; \delta_{\mathrm{H}}^{-1} B_*)$, which automatically satisfies $\gamma_0 \subseteq \{\varphi = 1\} \subseteq \{\varphi \ge M_*\}$. If $B_* \cap B(z, R) \neq \emptyset$, then $\gamma_{\max} = \gamma_0$ is enough. Suppose $B_* \cap B(z, R) = \emptyset$. Since φ is p-harmonic in $\delta_{\mathrm{H}}^{-1} B_* \setminus B(z, R)$, an application of the maximum principle yields a path $\gamma_1 \in \mathsf{Path}(\partial_i B_*, \partial B(z, R); \delta_{\mathrm{H}}^{-1} B_*)$ satisfying $\gamma_1 \subseteq \{\varphi \ge M_*\}$. Let us denote the endpoint of γ_1 in $\partial B(z, R)$ by x_1. By choosing $x_0 \in \partial_i B(z, R) \cap \delta_{\mathrm{H}}^{-1} B_*$ so that $\{x_0, x_1\} \in E$, we obtain the desired path $\gamma_{\max}$ by concatenating $\gamma_0, \{x_0, x_1\}$ and γ_1.

Using these paths $\gamma_{\min}$ and $\gamma_{\max}$, we can cary out the same argument as in the proof of Theorem 5.4. Indeed, since φ is positive and p-superharmonic in D, the log-Caccioppoli inequality (Lemma 5.3) yields

$$\mathcal{E}_{p,B_*}^{G}(\log \varphi) \le C_p \operatorname{cap}_p^G\bigl(B_*, \bigl(\delta_{\mathrm{H}}^{-1} B_*\bigr)^c\bigr).$$

Similar to Theorem 5.4, we can obtain $\max_{B_*} \varphi \le C_{\mathrm{H}} \min_{B_*} \varphi$, where C_{H} is the constant in Theorem 5.4. The desired estimate (5.10) follows from the above Harnack inequality using the standard Moser's oscillation lemma argument similar to Corollary 5.5.

Case 3: $\delta_{\mathrm{H}}^{-1} B_* \cap D^c \neq \emptyset$. A similar argument as Case 2 considering $1 - \varphi$ instead of φ gives the desired Hölder regularity (5.10), and the proof is completed. □

6 Sobolev space via a sequence of discrete energies

We consider a sequence of finite graphs that can be regarded as approximations of a metric space on a sequence of increasingly finer scales. The Sobolev space on a metric space is then defined using this sequence of discrete energies.

6.1 Approximating a metric space by a sequence of graphs

We introduce our assumptions on a sequence of graphs.

Definition 6.1. Let $\{\mathbb{G}_n = (V_n, E_n)\}_{n\in\mathbb{N}}$ be a sequence of finite, connected simple non-directed graphs. We say that a family of surjective maps $\{\pi_{n,k}\colon V_n \to V_k \mid 1 \le k < n, (n,k) \in \mathbb{N}^2\}$ is *projective* if $\pi_{n,k}$ is surjective for all $k < n$ and

$$\pi_{l,k} \circ \pi_{n,l} = \pi_{n,k}, \quad \text{for all } k < l < n \text{ with } k,l,n \in \mathbb{N}.$$

Given $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ and a projective family of maps $\{\pi_{n,k} : k < n\}$, we say that a sequence of probability measures $\{m_n \in \mathcal{P}(V_n)\}_{n\in\mathbb{N}}$, where $\mathcal{P}(V_n)$ denotes the set of probability measure on V_n, is *consistent* if

$$(\pi_{n,k})_* m_n = m_k \quad \text{for all } k < n.$$

Given a sequence of finite connected graphs $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$, a projective family of maps $\{\pi_{n,k} \mid k < n\}$, and a consistent family of probability measures $\{m_n\}_{n\in\mathbb{N}}$, we say that a sequence of functions $\{f_n\colon V_n \to \mathbb{R}\}_{n\in\mathbb{N}}$ is *conditional* with respect to $\{m_n\}_{n\in\mathbb{N}}$ if

$$f_k(v) = \frac{1}{m_k(v)} \sum_{w\in\pi_{n,k}^{-1}(\{v\})} f_n(w) m_n(w) \quad \text{for all } k < n, v \in V_k. \tag{6.1}$$

In the above definition, the graphs $\mathbb{G}_n$ can be regarded as approximating a metric space (K,d) at a sequence of increasingly finer scales, while the measures m_n can be considered to approximate a measure m on K. A conditional sequence of functions can be considered to approximate a function f on the metric space (K,d).

The sequence of measures $\{m_n\}_{n\in\mathbb{N}}$ in the above definition is often assumed to satisfy the condition given by the following definition.

Definition 6.2. Let $\{m_n \in \mathcal{P}(V_n)\}_{n\in\mathbb{N}}$ be a sequence of probability measures on a family of finite sets V_n. We say that such a sequence $\{m_n\}_{n\in\mathbb{N}}$ is *roughly uniform* if there exists $C_{\mathrm{u}} \ge 1$ such that

$$C_{\mathrm{u}}^{-1} m_n(v) \le \frac{1}{\# V_n} \le C_{\mathrm{u}} m_n(v), \quad \text{for all } n \in \mathbb{N}, v \in V_n. \tag{6.2}$$

We introduce a geometric condition on the sequence of graphs which relates different graphs in the sequence. Roughly speaking, the following condition states that $\operatorname{diam}(\mathbb{G}_n)$ grows like R_*^n and $\pi_{n+k,k}^{-1}(w)$ are ‘roundish’ in an uniform fashion; that is $\pi_{n+k,k}^{-1}(w)$ behave like balls in the graph $\mathbb{G}_{n+k}$ for all $w \in V_k$.

Definition 6.3. Let $R_* \in (1,\infty)$, let $\{\mathbb{G}_n = (V_n, E_n)\}_{n\in\mathbb{N}}$ be a sequence of finite, connected simple non-directed graphs, and let $\{\pi_{n,k}\colon V_n \to V_k \mid 1 \le k < n\}$ be a family of projective maps. We say that the sequence of graphs $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ equipped with the projective maps $\{\pi_{n,k}\colon V_n \to V_k \mid k < n\}$ is R_*-*scaled* if there exist $A_1, A_2 \in (1,\infty)$ so that the following holds: for any $n,k \in \mathbb{N}$, for all $w \in V_k$, there exists $c_n(w) \in V_{n+k}$ such that

$$B_{d_{n+k}}(c_n(w), A_1^{-1} R_*^n) \subset \pi_{n+k,k}^{-1}(w) \subset B_{d_{n+k}}(c_n(w), A_1 R_*^n) \tag{6.3}$$

and

$$d_{n+k}(c_n(w), c_n(w')) \leq A_2 R_*^n \quad \text{whenever } w, w' \in V_k \text{ satisfy } d_k(w, w') = 1, \tag{6.4}$$

where d_n denotes the graph distance of $\mathbb{G}_n$.

We next discuss discrete approximations of a metric space. Any compact metric space can be approximated by a sequence of graphs on increasing finer scales. This idea is present in various (closely related) notions such as hyperbolic filling [BBS22, BP03, BS18, BS, Ele97], K-approximation [BK02], quasi-visual approximation [BM22], generalized dyadic cubes [Chi90, Dav88, HK12, KRS12, Sas23], and partitions of a metric space indexed by tree [Kig20]. The following definition describes yet another way in which a sequence of graphs 'approximate' a compact metric space.

Definition 6.4 (compatibility)**.** Consider a compact metric space (K, d) and let $R_* \in (1, \infty), \theta \in (0, 1]$. Let $\{\mathbb{G}_n = (V_n, E_n)\}_{n \in \mathbb{N}}$ be a sequence of finite, connected simple non-directed graphs and let $\{\pi_{n,k} \colon V_n \to V_k \mid 1 \leq k < n\}$ be a family of projective maps. Let $d_n \colon V_n \times V_n \to \mathbb{Z}_{\geq 0}, n \in \mathbb{N}$ denote the corresponding graph metrics. We say that $\{\mathbb{G}_n\}$ along with $\{\pi_{n,k} : V_n \to V_k \mid 1 \leq k < n\}$ is R_*-*compatible* with (K, d) if there exists a sequence of maps $\{p_n \colon V_n \to K\}_{n \in \mathbb{N}}$, a collection of Borel set $\left\{\widetilde{K}_v \mid v \in V_n, n \in \mathbb{N}\right\}$ and $C \in [1, \infty)$ such that the following hold:

(i) (comparison of metrics)

$$C^{-1} \frac{d_n(x, y)}{R_*^n} \leq d(p_n(x), p_n(y)) \leq C \frac{d_n(x, y)}{R_*^n} \tag{6.5}$$

for all $x, y \in V_n$ and for all $n \in \mathbb{N}$.

(ii) (partition) For all $n \in \mathbb{N}$, the collection of sets $\left\{\widetilde{K}_v\right\}_{v \in V_n}$ form a partition of K; that is $\bigcup_{v \in V_n} \widetilde{K}_v = K$ and $\widetilde{K}_u \cap \widetilde{K}_w = \emptyset$ for all $u, w \in V_n$ with $u \neq w$.

(iii) (compatibility with projections) For all $1 \leq k < n$ and for all $v \in V_k$, we have

$$\widetilde{K}_v = \bigcup_{w \in \pi_{n,k}^{-1}(v)} \widetilde{K}_w. \tag{6.6}$$

(iv) (roundness of partition) For all $n \in \mathbb{N}, v \in V_n$, we have

$$B_d(p_n(v), C^{-1} R_*^{-n}) \subset \widetilde{K}_v \subset B_d(p_n(v), C R_*^{-n}). \tag{6.7}$$

Note that (6.5) implies that the points $\{p_n(v) \mid v \in V_n\}$ are $C^{-1} R_*^{-n}$-separated and that $\operatorname{diam}(V_n, d_n) \asymp R_*^n$.

We introduce a uniform notion of AR(d_{f}) for a sequence of graphs.

Definition 6.5. We shall say that the sequence $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ satisfies d_{f}*-Ahlfors regularity condition uniformly*, U-AR(d_{f}) for short, if there exists $C_{\mathrm{AR}} \geq 1$ such that for all $n \in \mathbb{N}$, $x \in V_n$, $R \in [1, \mathrm{diam}(\mathbb{G}_n) + 1)$,

$$C_{\mathrm{AR}}^{-1} R^{d_{\mathrm{f}}} \leq \#B_{d_n}(x, R) \leq C_{\mathrm{AR}} R^{d_{\mathrm{f}}}. \tag{U-AR(d_{f})}$$

The following elementary lemma explains the relationship between a metric space and a sequence of graphs approximating it in the sense of Definition 6.4 and the notions in Definitions 6.1 and 6.2.

Lemma 6.6. *Let (K, d) be a compact metric space and let m be a d_{f}-Ahlfors regular probability measure on (K, d). Let $\{\mathbb{G}_n = (V_n, E_n)\}_{n\in\mathbb{N}}$ be a sequence of connected, finite graphs and let $\{\pi_{n,k} : V_n \to V_k \mid 1 \leq k < n\}$ be a projective family of maps. Suppose that $\{\mathbb{G}_n\}$ along with $\{\pi_{n,k} \mid 1 \leq k < n\}$ is R_*-compatible with (K, d). Let $\{\widetilde{K}_v \in \mathcal{B}(K) \bigm| v \in V_n, n \in \mathbb{N}\}$ be a collection of Borel sets as given in Definition 6.4. Let*

$$m_n(v) \coloneqq m(\widetilde{K}_v)$$

for all $n \in \mathbb{N}, v \in V_n$. Then

(i) *The sequence of graphs $\{\mathbb{G}_n\}$ satisfies* U-AR(d_{f}).

(ii) *The family of measures $\{m_n\}$ is roughly uniform, and is consistent with respect to $\{\pi_{n,k} \mid 1 \leq k < n\}$.*

(iii) *For any $f \in L^1(K, m)$, the family of functions $M_n f : V_n \to \mathbb{R}$ defined by*

$$(M_n f)(v) = \frac{1}{m(\widetilde{K}_v)} \int_{\widetilde{K}_v} f\, dm, \quad \text{for all } n \in \mathbb{N}, v \in V_n, \tag{6.8}$$

is conditional with respect to $\{m_n\}$ and $\{\pi_{n,k} \mid 1 \leq k < n\}$.

The operator M_n converts a function on K to a function on V_n. We would sometimes like to construct functions on K using functions on V_n by defining

$$J_n f(\cdot) \coloneqq \sum_{v\in V_n} f(v) \mathbb{1}_{\widetilde{K}_v}(\cdot), \quad \text{for all } f : V_n \to \mathbb{R}, n \in \mathbb{N}. \tag{6.9}$$

6.2 Hypotheses on a sequence of graphs

A sequence of graphs approximating a metric space often satisfies some analytic properties in an uniform manner. To this end, we introduce uniform versions of *analytic conditions* such as $\mathrm{cap}_{p,\leq}(\beta)$, $\mathrm{BCL}_p(\zeta)$, and $\mathrm{PI}_p(\beta)$.

Definition 6.7. Let $\{\mathbb{G}_n = (V_n, E_n)\}_{n\in\mathbb{N}}$ be a sequence of graphs and let d_n be the graph metric of $\mathbb{G}_n$. Let $p \in (1, \infty)$, $d_{\mathrm{f}} > 0$, $\beta > 0$ and $\zeta \in \mathbb{R}$.

(1) We shall say that the sequence $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ satisfies *p-capacity upper bound with order β uniformly*, U-cap$_{p,\leq}(\beta)$ for short, if there exist $C_{\mathrm{cap}} > 0$ and $A_{\mathrm{cap}} \geq 1$ such that for any $n \in \mathbb{N}$, $x \in V_n$ and $R \in [1, \mathrm{diam}(\mathbb{G}_n)/A)$,

$$\mathrm{cap}_p^{G_n}\left(B_{d_n}(x,R), B_{d_n}(x,2R)^c\right) \leq C_{\mathrm{cap}} \frac{\#B_{d_n}(x,R)}{R^\beta}. \qquad (\text{U-cap}_{p,\leq}(\beta))$$

(2) We shall say that the sequence $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ satisfies *ball combinatorial p-Loewner property with order ζ uniformly*, U-BCL$_p(\zeta)$ for short, if there exists $A \geq 1$ such that the following hold: for any $\kappa > 0$ there exist $c_{\mathrm{BCL}}(\kappa) > 0$, $L_{\mathrm{BCL}}(\kappa) > 0$ such that

$$\mathrm{Mod}_p^{\mathbb{G}_n}(\{\theta \in \mathsf{Path}_{\mathbb{G}_n}(B_1, B_2) \mid \mathrm{diam}(\theta, d_n) \leq L_{\mathrm{BCL}}(\kappa)R\}) \geq c_{\mathrm{BCL}}(\kappa)R^\zeta \qquad (\text{U-BCL}_p(\zeta))$$

whenever $n \in \mathbb{N}$, $R \in [1, \mathrm{diam}(\mathbb{G}_n)/A)$ and B_i $(i = 1, 2)$ are balls in $\mathbb{G}_n$ with radii R satisfying $\mathrm{dist}_{d_n}(B_1, B_2) \leq \kappa R$. We also say that $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ satisfies U-BCL$_p^{\mathrm{low}}(\zeta)$ if $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ satisfies U-BCL$_p(\zeta)$ with $\zeta < 1$.

(3) We shall say that the sequence of graphs $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ satisfies *(p,p)-Poincaré inequality with order β uniformly*, U-PI$_p(\beta)$ for short, if there exist $C_{\mathrm{PI}}, A_{\mathrm{PI}} \geq 1$ such that for any $n \in \mathbb{N}, x \in V_n$, $R \geq 1$ and $f\colon V_n \to \mathbb{R}$,

$$\sum_{y \in B_{d_n}(x,R)} \left|f(y) - f_{B_{d_n}(x,R)}\right|^p \leq C_{\mathrm{PI}} R^\beta \mathcal{E}^{\mathbb{G}_n}_{p, B_{d_n}(x, A_{\mathrm{PI}}R)}(f). \qquad (\text{U-PI}_p(\beta))$$

Using the above definition, we can rephrase Theorem 4.2 for a sequence of graphs as follows.

Proposition 6.8. *Let $\{\mathbb{G}_n = (V_n, E_n)\}_{n\in\mathbb{N}}$ be a sequence of finite connected graphs. Let $p \in (1,\infty)$, $d_{\mathrm{f}} \geq 1$ and $\beta > 0$. Suppose that $\{\mathbb{G}_n\}$ satisfies* U-AR(d_{f}) *and* U-BCL$_p^{\mathrm{low}}(d_{\mathrm{f}} - \beta)$. *Then $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ satisfies* U-PI$_p(\beta)$ *(the associated constants $C_{\mathrm{PI}} > 0$ and $A_{\mathrm{PI}} \geq 1$ depend only on the constants involved in the assumptions).*

The following definition gives a uniform notion of the metric doubling property for a sequence of graphs.

Definition 6.9. Let $\{\mathbb{G}_n = (V_n, E_n)\}_{n\in\mathbb{N}}$ be a sequence of graphs and let d_n be the graph metric of $\mathbb{G}_n$.

(1) Define $L_* \coloneqq L_*(\{\mathbb{G}_n\}_{n\in\mathbb{N}}) \coloneqq \sup_{n\in\mathbb{N}} \deg(\mathbb{G}_n)$.

(2) We shall say that $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ is *uniformly metric doubling*, U-MD for short, if there exists $N_{\mathrm{D}} \geq 2$ such that given $n \in \mathbb{N}$, $x \in V_n$, $R \geq 1$ there exist $y_1, \dots, y_N \in V_n$ satisfying $B_{d_n}(x,R) \subseteq \bigcup_{i=1}^{N_{\mathrm{D}}} B_{d_n}(y_i, R/2)$.

Then the following property is an easy consequence of Remark 2.15.

Lemma 6.10. *Let $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ be a sequence of graphs satisfying U-AR(d_{f}) for some $d_{\mathrm{f}} > 0$. Then $L_* < \infty$ and $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ is U-MD. In addition, the doubling constant N_{D} can be chosen so that N_{D} depends only on C_{AR}.*

In order to state a version of Theorem 5.6 for a sequence of graphs, we introduce the following definition.

Definition 6.11. Let $\{\mathbb{G}_n = (V_n, E_n)\}_{n\in\mathbb{N}}$ be a sequence of finite, connected graphs. Let $p \in (1,\infty), \beta > 0, \vartheta \in (0,1]$. We say that the sequence of graphs $\{\mathbb{G}_n\}$ satisfies U-CF$_p(\vartheta,\beta)$ if there exists $C_* \in (0,\infty)$ so that the following holds: for all $n \in \mathbb{N}, v \in V_n, R \geq 1$ there exists $\varphi_{v,R} \colon V_n \to [0,1]$, so that

$$\varphi_{v,R}\big|_{B_{d_n}(v,R)} \equiv 1, \quad \mathrm{supp}[\varphi_{v,R}] \subseteq B_{d_n}(v,2R) \tag{6.10}$$

$$\mathcal{E}_p^{\mathbb{G}_n}(\varphi_{v,R}) \leq C_* \frac{\# B_{d_n}(v,R)}{R^\beta}, \tag{6.11}$$

$$|\varphi_{v,R}(x) - \varphi_{v,R}(y)| \leq C_* \left(\frac{d_n(x,y)}{R}\right)^\vartheta \quad \text{for all } x,y \in V_n. \tag{6.12}$$

The next result provides a family of Hölder continuous cutoff functions whose energies are controlled in a uniform manner. This is an immediate consequence of Theorem 5.6.

Proposition 6.12. *Let $\{\mathbb{G}_n = (V_n, E_n)\}_{n\in\mathbb{N}}$ be a sequence of finite connected graphs. Let $p \in (1,\infty), d_{\mathrm{f}} \geq 1$ and $\beta > 0$. Suppose that $\{\mathbb{G}_n\}$ satisfies U-AR(d_{f}), U-BCL$_p^{\mathrm{low}}(d_{\mathrm{f}} - \beta)$ and U-cap$_{p,\leq}(\beta)$. Then $\{\mathbb{G}_n\}$ satisfies U-CF$_p(\vartheta,\beta)$ (the associated constants $C_*, \vartheta > 0$ depend only on the constants involved in the assumptions).*

We would like to define p-energy as limit of re-scaled discrete energies. The re-scaling factor for discrete energies is suggested by the following weak monotonicity result for scaled discrete energies.

Theorem 6.13. *Let $\{\mathbb{G}_n = (V_n, E_n)\}_{n\in\mathbb{N}}$ be a sequence of finite, connected simple non-directed graphs equipped with the projective maps $\{\pi_{n,k} \colon V_n \to V_k; k < n\}$ and let $\{m_n \in \mathcal{P}(V_n)\}_{n\in\mathbb{N}}$ be a consistent sequence of probability measures. Suppose that $\{\mathbb{G}_n\}$ along with $\{\pi_{n,k}; k < n\}$ is R_*-scaled for some $R_* \in (1,\infty)$ and the sequence $\{m_n\}$ is roughly uniform. Let $p \in (1,\infty), d_{\mathrm{f}} \geq 1, \beta > 0$ and we further suppose that the sequence $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ satisfies U-AR(d_{f}) and U-PI$_p(\beta)$. There exists $C_{\mathrm{WM}} \in (1,\infty)$ depending only on the constants associated to the assumptions such that for any conditional sequence of functions $\{f_n \colon V_n \to \mathbb{R}\}_{n\in\mathbb{N}}$ with respect to $\{m_n\}$ and $\{\pi_{n,k}\}$, we have*

$$\mathcal{E}_p^{\mathbb{G}_k}(f_k) \leq C_{\mathrm{WM}} R_*^{l(\beta - d_{\mathrm{f}})} \mathcal{E}_p^{\mathbb{G}_{k+l}}(f_{k+l}) \quad \textit{for all } k,l \in \mathbb{N}. \tag{6.13}$$

Proof. Let $f_n \colon V_n \to \mathbb{R}, n \in \mathbb{N}$ denote an arbitrary conditional sequence of functions as above. Let $A_1, A_2 \in (1,\infty)$ be the constants as given in Definition 6.3, $C_u \in (1,\infty)$ be the constant in Definition 6.2. Set $A_3 = 2A_1 + A_2$. For any $v, w \in V_k$ such that $d_k(v,w) = 1$, we have

$$\pi_{k+l,k}^{-1}(v) \cup \pi_{k+l,k}^{-1}(w) \subset B_{d_{k+l}}(c_l(v), A_3 R_*^l) \quad \text{(by (6.3) and (6.4))}. \tag{6.14}$$

There is $C_1 \in [1,\infty)$ depending only on the constants involved in U-AR(d_{f}), roughly uniform, and R_*-scaled properties such that

$$C_1^{-1}R_*^{-nd_{\mathrm{f}}} \le m_n(v) \le C_1 R_*^{-nd_{\mathrm{f}}} \quad \text{for all } n\in\mathbb{N}, v\in V_n. \tag{6.15}$$

For any $v,w\in V_k$ such that $d_k(v,w)=1$ and for all $\alpha\in\mathbb{R}$, we have

$$\begin{aligned}
|f_k(v)-f_k(w)| &\le |f_k(v)-\alpha| + |f_k(w)-\alpha| \\
&\le \left|\sum_{v_1\in\pi_{k+l,k}^{-1}(v)} f_{k+l}(v_1)\frac{m_{k+l}(v_1)}{m_k(v)} - \alpha\right| + \left|\sum_{w_1\in\pi_{k+l,k}^{-1}(w)} f_{k+l}(w_1)\frac{m_{k+l}(w_1)}{m_k(w)} - \alpha\right| \\
&\le \sum_{v_1\in\pi_{k+l,k}^{-1}(v)} \frac{m_{k+l}(v_1)}{m_k(v)}|f_{k+l}(v_1)-\alpha| + \sum_{w_1\in\pi_{k+l,k}^{-1}(w)} \frac{m_{k+l}(w_1)}{m_k(w)}|f_{k+l}(w_1)-\alpha| \\
&\overset{(6.15)}{\le} C_1^2 R_*^{-ld_{\mathrm{f}}}\left(\sum_{v_1\in\pi_{k+l,k}^{-1}(v)}|f_{k+l}(v_1)-\alpha| + \sum_{w_1\in\pi_{k+l,k}^{-1}(w)}|f_{k+l}(w_1)-\alpha|\right) \\
&\overset{(6.14)}{\le} 2C_1^2 R_*^{-ld_{\mathrm{f}}} \sum_{v_1\in B_{d_{k+l}}(c_l(v),A_3R_*^l)} |f_{k+l}(v_1)-\alpha| \\
&\lesssim \frac{1}{\#B_{d_{k+l}}(c_l(v),A_3R_*^l)} \sum_{v_1\in B_{d_{k+l}}(c_l(v),A_3R_*^l)} |f_{k+l}(v_1)-\alpha|,
\end{aligned} \tag{6.16}$$

where in the last line, we used the U-AR(d_{f}). Let us choose $\alpha = (f_{k+l})_{B_{d_{k+l}}(c_l(v),A_3R_*^l)}$ in (6.16) and use Poincaré inequality U-PI$_p(\beta)$ to obtain

$$\begin{aligned}
|f_k(v)-f_k(w)|^p &\lesssim \frac{1}{\#B_{d_{k+l}}(c_l(v),A_3R_*^l)} \sum_{v_1\in B_{d_{k+l}}(c_l(v),A_3R_*^l)} \left|f_{k+1}(v_1) - (f_{k+1})_{B_{d_{k+l}}(c_l(v),A_3R_*^l)}\right|^p \\
&\lesssim \frac{R_*^{l\beta}}{\#B_{d_{k+l}}(c_l(v),A_3R_*^l)} \mathcal{E}_{p,B_{d_{k+l}}(c_l(v),A_{\mathrm{PI}}A_3R_*^l)}^{\mathbb{G}_{k+l}}(f_{k+l}) \quad (\text{by U-PI}_p(\beta)) \\
&\lesssim R_*^{l(\beta-d_{\mathrm{f}})} \mathcal{E}_{p,B_{d_{k+l}}(c_l(v),A_{\mathrm{PI}}A_3R_*^l)}^{\mathbb{G}_{k+l}}(f_{k+l})
\end{aligned} \tag{6.17}$$

for any $v,w\in V_k$ such that $d_k(v,w)=1$. Using Lemma 6.10, we obtain

$$\mathcal{E}_p^{\mathbb{G}_k}(f_k) = \sum_{\{v,w\}\in E_k} |f_k(v)-f_k(w)|^p \overset{(6.17)}{\lesssim} R_*^{l(\beta-d_{\mathrm{f}})} \sum_{v\in V_k} \mathcal{E}_{p,B_{d_{k+l}}(c_l(v),A_{\mathrm{PI}}A_3R_*^l)}^{\mathbb{G}_{k+l}}(f_{k+l}). \tag{6.18}$$

By (6.3), the points $\{c_l(v) \mid v\in V_k\}$ are $2A_1^{-1}R_*^l$-separated for all $k,l\in\mathbb{N}$. Since $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ are U-MD by Lemma 6.10, there exists $C_2>1$ (depending only on $A_{\mathrm{PI}}, A_1, A_2$ and the constants involved in U-AR(d_{f})) such that

$$\sum_{v\in V_k} \mathbb{1}_{B_{d_{k+l}}(c_l(v),A_{\mathrm{PI}}A_3R_*^l)} \le C_2, \quad \text{for all } k,l\in\mathbb{N}. \tag{6.19}$$

The desired estimate (6.13) follows immediately from (6.18) and (6.19). □

Remark 6.14. In the work [Kig23], the notion of *conductive homogeneity* plays an important role to develop the theory of $(1,p)$-Sobolev spaces via discretizations. The estimate (6.17) can be regarded as a variant of this condition.

6.3 Sobolev space and cutoff functions

We now explain our strategy to construct p-energy as a scaling limit of discrete p-energies in a general setting. The following assumption guarantees that our Sobolev space satisfies good properties.

Assumption 6.15. Let $p \in (1,\infty)$, $d_{\mathrm{f}} \in [1,\infty)$, $\beta > 0$ and $\vartheta \in (0,1]$. Let (K,d) be a connected compact metric space with $\#K \geq 2$ and let m be a d_{f}-Ahlfors regular probability measure on (K,d). Let $\{\mathbb{G}_n = (V_n, E_n)\}_{n\in\mathbb{N}}$ be a sequence of finite, connected simple non-directed graphs and let $\{\pi_{n,k} \mid 1 \leq k < n\}$ denote a projective family of maps. There exists $R_* \in (1,\infty)$ such that $\{\mathbb{G}_n\}$ along with $\{\pi_{n,k}\}$ is R_*-scaled and R_*-compatible with (K,d). Furthermore, $\{\mathbb{G}_n\}$ satisfies U-PI$_p(\beta)$ and U-CF$_p(\vartheta,\beta)$.

The weak monotonicity of discrete energies (Theorem 6.13) suggests the following definition of Sobolev space.

Definition 6.16. Under the setting of Assumption 6.15, we define the normalized energy of $f \in L^p(K,m)$ for any $n \in \mathbb{N}$ and $A \subseteq V_n$ as

$$\widetilde{\mathcal{E}}_{p,A}^{(n)}(f) \coloneqq R_*^{n(\beta-d_{\mathrm{f}})}\mathcal{E}_{p,A}^{\mathbb{G}_n}(M_n f), \tag{6.20}$$

where $M_n f$ is as given in (6.8). For ease of notation, we set $\widetilde{\mathcal{E}}_p^{(n)}(f) \coloneqq \widetilde{\mathcal{E}}_{p,V_n}^{(n)}(f)$. Define our $(1,p)$-Sobolev space $\mathcal{F}_p(K,d,m)$ by

$$\mathcal{F}_p(K,d,m) \coloneqq \left\{ f \in L^p(K,m) \;\middle|\; \sup_{n\in\mathbb{N}} \widetilde{\mathcal{E}}_p^{(n)}(f) < \infty \right\}. \tag{6.21}$$

We also set $|f|_{\mathcal{F}_p(K,d,m)} \coloneqq \left(\sup_{n\in\mathbb{N}} \widetilde{\mathcal{E}}_p^{(n)}(f)\right)^{1/p}$ and $\|f\|_{\mathcal{F}_p(K,d,m)} \coloneqq \|f\|_{L^p(m)} + |f|_{\mathcal{F}_p(K,d,m)}$. We use $\mathcal{F}_p$ instead of $\mathcal{F}_p(K,d,m)$ when no confusion can occur.

Hereafter in this section, we always assume that Assumption 6.15 holds. Thanks to Theorem 6.13 and Lemma 6.6, we have

$$\liminf_{n\to\infty} \widetilde{\mathcal{E}}_p^{(n)}(f) \asymp \limsup_{n\to\infty} \widetilde{\mathcal{E}}_p^{(n)}(f) \asymp \sup_{n\in\mathbb{N}} \widetilde{\mathcal{E}}_p^{(n)}(f), \quad \text{for all } f \in L^p(K,m). \tag{6.22}$$

In particular,

$$\mathcal{F}_p = \left\{ f \in L^p(K,m) \;\middle|\; \liminf_{n\to\infty} \widetilde{\mathcal{E}}_p^{(n)}(f) < \infty \right\} = \left\{ f \in L^p(K,m) \;\middle|\; \limsup_{n\to\infty} \widetilde{\mathcal{E}}_p^{(n)}(f) < \infty \right\}.$$

Some properties of $\mathcal{F}_p$ are already mentioned in [Kig23, Section 3.2] in the framework of weighted partition theory developed in [Kig20]. We summarize the basic properties of the Sobolev space $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ in the following theorem.

Theorem 6.17. *Let (K, d) be a connected compact metric space with a d_f-Ahlfors regular probability measure m and let $\{\mathbb{G}_n = (V_n, E_n)\}_{n\in\mathbb{N}}$ be a sequence of finite connected graphs satisfying Assumption 6.15. Let $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ denote the normed linear space in Definition 6.16. Then $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ satisfies the following properties.*

(i) *$(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ is a Banach space.*

(ii) *$(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ admits an equivalent uniformly convex norm. In particular, $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ is a reflexive Banach space.*

(iii) *The Banach space $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ is separable.*

(iv) *$\mathcal{F}_p \cap \mathcal{C}(K)$ is dense in $\mathcal{C}(K)$ with respect to the uniform norm.*

(v) *$\mathcal{F}_p \cap \mathcal{C}(K)$ is dense in the Banach space $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$.*

The combination of properties (iv) and (v) is referred to as *regularity* in the theory of Dirichlet forms [FOT, p. 6]. The proof of Theorem 6.17 will be completed over this section and the next. It is easy to show the completeness of $\mathcal{F}_p$.

Proof of Theorem 6.17(i). Let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence in $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$. Since the convergence in $\mathcal{F}_p$ implies the convergence in L^p, the sequence $\{f_n\}_{n\geq 1}$ converges in L^p to some $f \in L^p(K, m)$. By the dominated convergence theorem, for any $k \in \mathbb{N}$ and $w \in V_k$, we have $M_k f_n(w) \to M_k f(w)$ as $n \to \infty$. Also, since $\{f_n\}_{n\geq 1}$ is a Cauchy sequence in $\mathcal{F}_p$, for any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\sup_{n\wedge l\geq N(\varepsilon)} \sup_{k\in\mathbb{N}} \widetilde{\mathcal{E}}_p^{(k)}(f_n - f_l) \leq \varepsilon.$$

Letting $l \to \infty$ in the estimate $\widetilde{\mathcal{E}}_p^{(k)}(f_n - f_l) \leq \varepsilon$ and taking the supremum over $k \in \mathbb{N}$ and $n \geq N(\varepsilon)$, we obtain

$$\sup_{n\geq N(\varepsilon)} \sup_{k\in\mathbb{N}} \widetilde{\mathcal{E}}_p^{(k)}(f_n - f) \leq \varepsilon. \tag{6.23}$$

Therefore, for any $k \in \mathbb{N}$,

$$\widetilde{\mathcal{E}}_p^{(k)}(f)^{1/p} \leq \widetilde{\mathcal{E}}_p^{(k)}(f_{N(\varepsilon)} - f)^{1/p} + \widetilde{\mathcal{E}}_p^{(k)}(f_{N(\varepsilon)})^{1/p} \leq \varepsilon^{1/p} + \sup_{n\geq 1} |f_n|_{\mathcal{F}_p}.$$

This implies $|f|_{\mathcal{F}_p} \leq \sup_{n\geq 1} |f_n|_{\mathcal{F}_p} < \infty$ and thus $f \in \mathcal{F}_p$. The required convergence $f_n \to f$ in $\mathcal{F}_p$ is also deduced from the L^p-convergence of f_n and (6.23). □

Next, we will prove *reflexivity* and *separability* of the Banach space $\mathcal{F}_p$. The reflexivity of such a function space is proved by the second-named author in [Shi24] by showing the existence a comparable *uniformly convex* norm. To construct a uniformly convex norm on $\mathcal{F}_p$ which is equivalent to $\|\cdot\|_{\mathcal{F}_p}$, we need the notion of Γ-*convergence*; see [Dal] for details. We first recall the definition.

Definition 6.18 ([Dal, Definition 4.1 and Proposition 8.1])**.** Let X be a first-countable topological space and let $F\colon X \to \mathbb{R} \cup \{\pm\infty\}$. A sequence of functionals $\{F_n\colon X \to \mathbb{R} \cup \{\pm\infty\}\}_{n\in\mathbb{N}}$ Γ-*converges* to F if the following conditions hold for any $x \in X$:

(i) (liminf inequality) If $x_n \to x$ in X, then $F(x) \le \liminf_{n\to\infty} F_n(x_n)$.

(ii) (limsup inequality) There exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that

$$x_n \to x \text{ in } X \quad \text{and} \quad \limsup_{n\to\infty} F_n(x_n) \le F(x). \tag{6.24}$$

A sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying (6.24) is called a *recovery sequence of* $\{F_n\}_{n\in\mathbb{N}}$ *at* x.

The following compactness result is useful to construct Γ-limits.

Proposition 6.19 ([Dal, Theorem 8.5])**.** *Suppose that X is a topological space with a countable base. Then any sequence of functionals $\{F_n\colon X \to \mathbb{R} \cup \{\pm\infty\}\}_{n\in\mathbb{N}}$ has a Γ-convergent subsequence.*

Now we can establish the reflexivity of $\mathcal{F}_p$.

Proof of Theorem 6.17(ii). This is essentially the same as in [Shi24, Theorem 5.9], so we briefly outline the proof. By Proposition 6.19, we have a Γ-cluster point E_p of the sequence of functionals $\bigl\{\widetilde{\mathcal{E}}_p^{(n)}\bigr\}_{n\in\mathbb{N}}$ on $L^p(K,m)$. It is easy to show that $\mathsf{E}_p(\,\cdot\,)^{1/p}$ is a semi-norm on $\mathcal{F}_p$. The liminf inequality implies $\mathsf{E}_p(\,\cdot\,)^{1/p} \le |\cdot|_{\mathcal{F}_p}$. A combination of limsup inequality and weak monotonicity (Theorem 6.13) implies the converse estimate $\mathsf{E}_p(\,\cdot\,)^{1/p} \gtrsim |\cdot|_{\mathcal{F}_p}$. Hence $|||f||| := \bigl(\|f\|_{L^p}^p + \mathsf{E}_p(f)\bigr)^{1/p}$ is a norm on $\mathcal{F}_p$, which is equivalent to $\|\cdot\|_{\mathcal{F}_p}$. Noting that $|||\cdot|||$ is a Γ-cluster point of $\|\cdot\|_{p,n} := \Bigl(\|\cdot\|_{L^p}^p + \widetilde{\mathcal{E}}_p^{(n)}(\,\cdot\,)\Bigr)^{1/p}$, which can be regarded as the L^p-norm on $K \sqcup E_n$, we easily obtain p-Clarkson's inequality of $|||\cdot|||$:

$$\begin{cases} |||f+g|||^{p/(p-1)} + |||f-g|||^{p/(p-1)} \le 2\bigl(|||f|||^p + |||g|||^p\bigr)^{1/(p-1)} & \text{if } p \le 2, \\ |||f+g|||^p + |||f-g|||^p \le 2\bigl(|||f|||^{p/(p-1)} + |||g|||^{p/(p-1)}\bigr)^{p-1} & \text{if } p \ge 2. \end{cases} \tag{6.25}$$

Then $(\mathcal{F}_p, |||\cdot|||)$ is a uniform convex Banach space [Cla36, p. 403], so the Milman–Pettis theorem implies the reflexivity of $\mathcal{F}_p$. □

In [Shi24, Theorem 5.10], the separability of $\mathcal{F}_p$ has shown by using its reflexivity in the situation that $\mathcal{F}_p$ is continuously embedded into $\mathcal{C}(K)$ (cf. [Kig23, Theorem 3.22] or [Shi24, Theorem 5.1]). The proof of [Shi24, Theorem 5.10] essentially relies on such an embedding. Here, we will adopt another simple way to show the separability by using an idea in [AHM23].

Proof of Theorem 6.17(iii). The Banach space $\mathcal{F}_p$ is reflexive by Theorem 6.17(ii), and $L^p(K,m)$ is separable since K is separable. Clearly, the identity map $i\colon \mathcal{F}_p \to L^p(K,m)$ is a bounded linear injective map, so $\mathcal{F}_p$ is separable by [AHM23, Proposition 4.1]. □

We will next show the density of $\mathcal{F}_p \cap \mathcal{C}(K)$ in $\mathcal{C}(K)$ with respect to the uniform norm. To show such the density, a standard idea is to use Stone–Weierstrass theorem by showing that $\mathcal{F}_p \cap \mathcal{C}(K)$ is an algebra that separates points of K. We recall Arzelá–Ascoli type theorem for (possibly) discontinuous functions in order to construct a function in $\mathcal{F}_p \cap \mathcal{C}(K)$ that separates two distinct points (a cutoff function). The proof that $\mathcal{F}_p \cap \mathcal{C}(K)$ is an algebra will be done in the next subsection.

Lemma 6.20. *Let (X, d) be a totally bounded metric space. Let $u_n \colon X \to \mathbb{R}, n \in \mathbb{N}$ be a sequence of functions. Assume that there exist a non-decreasing function $\eta \colon [0,\infty) \to [0,\infty)$ and a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ of non-negative numbers such that $\lim_{t\downarrow 0}\eta(t) = 0$, $\lim_{n\to\infty}\delta_n = 0$, $\sup_{n\in\mathbb{N}, x\in X}|u_n(x)| < \infty$ and*

$$|u_n(x) - u_n(y)| \le \eta(\mathsf{d}(x,y)) + \delta_n \quad \text{for all } x, y \in X \text{ and } n \in \mathbb{N}. \tag{6.26}$$

Then there exist a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ and $u \in \mathcal{C}(X)$ with

$$|u(x) - u(y)| \le \eta(\mathsf{d}(x,y)) \quad \text{for all } x, y \in X,$$

such that $\sup_{x\in X}|u_{n_k}(x) - u(x)| \to 0$ as $k \to \infty$.

Proof. This is a simplified version of [Kig23, Lemma D.1]. Indeed, the case $(Y, d_Y) = (\mathbb{R}, |\cdot|)$ in [Kig23, Lemma D.1] is enough to obtain the required statement. □

The next proposition constructs cutoff functions with controlled energy in $\mathcal{F}_p \cap \mathcal{C}(K)$. We use the following useful notation. For $A \subseteq K$, we define

$$V_n(A) \coloneqq \bigl\{w \in V_n \bigm| \widetilde{K}_w \cap A \neq \emptyset\bigr\}. \tag{6.27}$$

Proposition 6.21. *There exists $C \in (1,\infty)$ depending only on the constants associated with Assumption 6.15 such that for any $r > 0, x \in K$ such that $B_d(x,2r) \neq K$, we have a function $\psi_{x,r} \in \mathcal{F}_p \cap \mathcal{C}(K)$ such that $\psi_{x,r}\big|_{B_d(x,r)} = 1$, $\operatorname{supp}[\psi_{x,r}] \subseteq B_d(x,2r)$ and*

$$\sup_{n\in\mathbb{N}} \widetilde{\mathcal{E}}_p^{(n)}(\psi_{x,r}) \le C r^{d_{\mathrm{f}}-\beta}.$$

Proof. Let $\{\widetilde{K}_v \mid v \in V_n, n \in \mathbb{N}\}, C \in (1,\infty)$ be as given in Definition 6.4. By (6.5) and (6.7), we have

$$\widetilde{K}_w \subset B_d(x, r + 2CR_*^{-n} + CRR_*^{-n}) \quad \text{for any } w \in \textstyle\bigcup_{v\in V_n(B_d(x,r))} B_{d_n}(v,R). \tag{6.28}$$

We choose $R_n > 0$ so that $CR_nR_*^{-n} = r/2$ and a maximal $R_n/2$-separated subset N of $V_n(B_d(x,r))$ with respect to the metric d_n, so that $\bigcup_{w\in N} B_{d_n}(w, R_n/2) \supset V_n(B_d(x,r))$. Since $\{p_n(w) \mid w \in N\}$ is $C^{-1}(R_n/2)R_*^{-n}$-separated and satisfies $\{p_n(w)\}_{w\in N} \subset B_d(x, r + CR_*^{-n})$. Therefore by the d_{f}-Ahlfors regularity of m, we obtain

$$\#N \lesssim \left(\frac{r + cR_*^{-n}}{R_nR_*^{-n}}\right)^{d_{\mathrm{f}}} \lesssim \left(\frac{R_nR_*^{-n} + R_*^{-n}}{R_nR_*^{-n}}\right)^{d_{\mathrm{f}}} \lesssim 1 \tag{6.29}$$

for all n large enough so that $R_n \geq 1$.

For n large enough so that $2CR_*^{-n} < r/2$, we have $R_n \geq 2$ and $\widetilde{K}_w \subset B_d(x, 2r)$ for any $w \in \bigcup_{v \in V_n(B_d(x,r))} B_{d_n}(v, R_n)$ (by (6.28)). Therefore by applying U-CF$_p(\vartheta, \beta)$, for each $w \in N$, there exists $\varphi_{w,R_n/2} \colon V_n \to [0,1]$ such that $\varphi_{w,R_n/2}\big|_{B_{d_n}(w,R_n/2)} \equiv 1$, $\operatorname{supp}[\varphi_{w,R_n/2}] \subseteq B_{d_n}(w, R_n)$,

$$\mathcal{E}_p^{\mathbb{G}_n}(\varphi_{w,R_n/2}) \lesssim R_n^{d_{\mathrm{f}}-\beta},$$

and $\varphi_{w,R_n/2}$ satisfies the Hölder regularity condition (6.12). Hence by (6.28) and (6.29), the function $\varphi_n \colon V_n \to \mathbb{R}$ defined by

$$\varphi_n \coloneqq \max_{w \in N} \varphi_{w,R_n/2}$$

satisfies $J_n\varphi_n\big|_{B_d(x,r)} \equiv 1$, $\operatorname{supp}_m[J_n\varphi_n] \subseteq B_d(x, 2r)$,

$$\varphi_n \equiv 1 \text{ on } V_n(B_d(x,r)), \quad \mathcal{E}_p^{\mathbb{G}_n}(\varphi_n) \lesssim R_n^{d_{\mathrm{f}}-\beta} \lesssim r^{d_{\mathrm{f}}-\beta} R_*^{n(d_{\mathrm{f}}-\beta)}, \tag{6.30}$$

and

$$|\varphi_n(v_1) - \varphi_n(v_2)| \lesssim \left(\frac{d_n(v_1, v_2)}{R_n}\right)^{\vartheta}, \quad \text{for all } v_1, v_2 \in V_n, \tag{6.31}$$

for all $n \in \mathbb{N}$ so that $2CR_*^{-n} < r/2$. To estimate the energy, we used the elementary inequality $\mathcal{E}_p^{\mathbb{G}_n}(\max_{w \in N} \varphi_{w,R_n/2}) \leq \sum_{w \in N} \mathcal{E}_p^{\mathbb{G}_n}(\varphi_{w,R_n/2})$ (see Lemma 2.6(a)). By Lemma 6.20, (6.31), (6.5), and (6.7), there exists a subsequence $\{J_{n_k}\varphi_{n_k}\}_k$ of $\{J_n\varphi_n\}_n$ which converges uniformly to a function $\psi_{x,r} \in \mathcal{C}(K)$. Then it is clear that $\psi_{x,r}\big|_{B_d(x,r)} \equiv 1$ and $\operatorname{supp}[\psi_{x,r}] \subseteq B_d(x, 2r)$. Using weak monotonicity (Theorem 6.13) and dominated convergence theorem, we obtain

$$\begin{aligned}\widetilde{\mathcal{E}}_p^{(n)}(\psi_{x,r}) &= R_*^{n(\beta-d_{\mathrm{f}})}\mathcal{E}_p^{\mathbb{G}_n}(M_n\psi_{x,r}) = \lim_{n_k \to \infty} R_*^{n(\beta-d_{\mathrm{f}})}\mathcal{E}_p^{\mathbb{G}_n}(M_n J_{n_k}\varphi_{n_k}) \\ &\overset{(6.13)}{\lesssim} \liminf_{n_k \to \infty} R_*^{n_k(\beta-d_{\mathrm{f}})}\mathcal{E}_p^{\mathbb{G}_{n_k}}(\varphi_{n_k}) \overset{(6.30)}{\lesssim} r^{d_{\mathrm{f}}-\beta}.\end{aligned}$$

Therefore $\psi_{x,r} \in \mathcal{F}_p \cap \mathcal{C}(K)$ and it satisfies the desired bound on energy. □

6.4 Scaling limit of discrete energies and regularity

In the rest of this section, we suppose that Assumption 6.15 holds as in the previous subsection. In this setting, we will construct an 'improved' p-energy type functionals on (K, d, m), which verifies that $\mathcal{F}_p \cap \mathcal{C}(K)$ is an algebra.

Theorem 6.22. *There exist a constant $C \geq 1$ (depending only on the constants associated with Assumption 6.15) and $\mathcal{E}_p^{\Gamma} \colon \mathcal{F}_p \to [0, \infty)$ such that the following hold:*

(i) $\mathcal{E}_p^{\Gamma}(\,\cdot\,)^{1/p}$ *is a semi-norm on* $\mathcal{F}_p$ *satisfying* $C^{-1}|f|_{\mathcal{F}_p} \leq \mathcal{E}_p^{\Gamma}(f)^{1/p} \leq |f|_{\mathcal{F}_p}$ *for all* $f \in \mathcal{F}_p$. *Moreover, it satisfies* p-Clarkson's inequality*: for any* $f, g \in \mathcal{F}_p$,

$$\begin{cases} \mathcal{E}_p^{\Gamma}(f+g)^{1/(p-1)} + \mathcal{E}_p^{\Gamma}(f-g)^{1/(p-1)} \leq 2\bigl(\mathcal{E}_p^{\Gamma}(f) + \mathcal{E}_p^{\Gamma}(g)\bigr)^{1/(p-1)} & \text{if } p \leq 2, \\ \mathcal{E}_p^{\Gamma}(f+g) + \mathcal{E}_p^{\Gamma}(f-g) \leq 2\bigl(\mathcal{E}_p^{\Gamma}(f)^{1/(p-1)} + \mathcal{E}_p^{\Gamma}(g)^{1/(p-1)}\bigr)^{p-1} & \text{if } p \geq 2, \end{cases}$$

In particular, $\mathcal{E}_p^{\Gamma}(\,\cdot\,)^{1/p}$ *is uniformly convex.*

(ii) *For any* $f \in \mathcal{F}_p$ *and* 1*-Lipschitz function* $\varphi \in \mathcal{C}(\mathbb{R})$*, we have* $\varphi \circ f \in \mathcal{F}_p$ *and* $\mathcal{E}_p^{\Gamma}(\varphi \circ f) \leq \mathcal{E}_p^{\Gamma}(f)$.

(iii) *If* $f, g \in \mathcal{F}_p \cap L^{\infty}(K, m)$*, then* $f \cdot g \in \mathcal{F}_p$ *and*

$$\mathcal{E}_p^{\Gamma}(f \cdot g) \leq c_p\bigl(\|f\|_{L^\infty}^p \vee \|g\|_{L^\infty}^p\bigr)\Bigl(\mathcal{E}_p^{\Gamma}(f) + \mathcal{E}_p^{\Gamma}(g)\Bigr). \tag{6.32}$$

where $c_p \in (0, \infty)$ *is a constant determined solely by* p.

(iv) $\mathcal{E}_p^{\Gamma}$ *is lower semi-continuous on* $L^p(K, m)$*. (Here we regard* $\mathcal{E}_p^{\Gamma}$ *as a* $[0, \infty]$*-valued functional by defining* $\mathcal{E}_p^{\Gamma}(f) \coloneqq \infty$ *for* $f \in L^p(K, m) \setminus \mathcal{F}_p$*.)*

(v) *Let* $T\colon (K, \mathcal{B}(K), m) \to (K, \mathcal{B}(K), m)$ *be a measurable map satisfying* $\widetilde{\mathcal{E}}_p^{(n)}(f \circ T) = \widetilde{\mathcal{E}}_p^{(n)}(f)$ *for any* $n \in \mathbb{N}$ *and* $f \in L^p(K, m)$*. Then* $f \circ T \in \mathcal{F}_p$ *for any* $f \in \mathcal{F}_p$ *and* $\mathcal{E}_p^{\Gamma}(f \circ T) = \mathcal{E}_p^{\Gamma}(f)$.

Remark. After submitting this work to the journal, the notion of the *generalized p-contraction property* was introduced in [KS24+b], which is a generalization of the properties (i)-(iii) above. One can show that $(\mathcal{E}_p^{\Gamma}, \mathcal{F}_p)$ satisfies the generalized p-contraction property ([KS24+b, Theorem 8.19 and Remark 8.20]). In particular, (6.32) can be updated as follows:

$$\mathcal{E}_p^{\Gamma}(f \cdot g)^{1/p} \leq \|g\|_{L^\infty}\, \mathcal{E}_p^{\Gamma}(f)^{1/p} + \|f\|_{L^\infty}\, \mathcal{E}_p^{\Gamma}(g)^{1/p};$$

see [KS24+b, Proposition 2.3(d)].

Proof. Let $\mathcal{E}_p^{\Gamma} = \mathsf{E}_p$ be a Γ-cluster point of $\bigl\{\widetilde{\mathcal{E}}_p^{(n)}\bigr\}_{n \in \mathbb{N}}$ as the proof of Theorem 6.17(ii). Then the properties in (i) except for p-Clarkson's inequality are already shown, and p-Clarkson's inequality of $\mathcal{E}_p^{\Gamma}$ follows from a similar argument as in Theorem 6.17(ii).

(ii) Once we know $\|f - f_n\|_{L^p} \to 0$ as $n \to \infty$ for any $f \in L^p(K, m)$ and $f_n \coloneqq J_n\bigl(M_n f\bigr)$, where $J_n\colon \mathbb{R}^{V_n} \to L^0(K, m)$ be the operator defined in (6.9), a straightforward modification of [Kig23, Theorem 3.21(b)] proves the assertion. So we will prove $\|f - f_n\|_{L^p} \to 0$ as $n \to \infty$. Note that $|M_n f(z)|^p \leq \fint_{\widetilde{K}_z} |f|^p\,dm$ for all $z \in V_n$ by Jensen's inequality, and hence $\|f_n\|_{L^p} \leq \|f\|_{L^p}$. For $x \in K$, let $z \in V_n$ be the unique element such that $x \in \widetilde{K}_z$. Then, by (6.7),

$$|f_n(x)| = |M_n f(z)| \leq \frac{m\bigl(B_d(x, 2CR_*^{-n})\bigr)}{m(\widetilde{K}_z)} \mathscr{M}|f|(x),$$

where $C \geq 1$ is the constant in (6.7) and $\mathscr{M}\colon L^p(K,m) \to L^p(K,m)$ is the *Hardy–Littlewood maximal operator*: $\mathscr{M}f(x) = \sup_{r>0} ⨍_{B_d(x,r)} |f(y)|\, m(dy)$ for $f \in L^p(K,m)$. Since m is Ahlfors regular, we have $\|\mathscr{M}f\|_{L^p} \lesssim \|f\|_{L^p}$ for any $f \in L^p(K,m)$ (see [HKST, Theorem 3.5.6] for example) and, by (6.7),

$$\sup_{n\in\mathbb{N}, z\in V_n, x\in K_z} \frac{m(B_d(x, 2CR_*^{-n}))}{m(\widetilde{K}_z)} < \infty. \tag{6.33}$$

Thus each f_n is dominated by $C'\mathscr{M}|f| \in L^p(K,m)$ for some constant $C' > 0$.

We next show m-a.e. convergence of $\{f_n\}_{n\in\mathbb{N}}$. Since m is Ahlfors regular, the Lebesgue differentiation theorem on (K,d,m) holds (see [HKST, Section 3.4] for example), i.e., the set $\mathscr{L}_f$ (*Lebesgue points of* f) defined by

$$\mathscr{L}_f := \left\{ x \in K \;\middle|\; \lim_{r\downarrow 0} ⨍_{B_d(x,r)} |f(x) - f(y)|\, m(dy) = 0 \right\}$$

is a Borel set and $m(K \setminus \mathscr{L}_f) = 0$. Let $x \in \mathscr{L}_f$ and let $z \in V_n$ be the unique element such that $x \in \widetilde{K}_z$. Then we see that

$$\begin{aligned} |f(x) - f_n(x)| &\leq ⨍_{\widetilde{K}_z} |f(x) - f(y)|\, m(dy) \\ &\leq \frac{m(B_d(x, 2CR_*^{-n}))}{m(\widetilde{K}_z)} ⨍_{B_d(x, 2CR_*^{-n})} |f(x) - f(y)|\, m(dy). \end{aligned}$$

By (6.33), we have $\lim_{n\to\infty} |f(x) - f_n(x)| = 0$ for all $x \in \mathscr{L}_f$. The dominated convergence theorem implies $\|f - f_n\|_{L^p} \to 0$.

(iii) The case $f = g$ easily follows from (ii) by considering a 1-Lipschitz function $\varphi \in \mathcal{C}(\mathbb{R})$ satisfying $\varphi(t) = (2\|f\|_{L^\infty})^{-1}t^2$ for any $t \in \left[-\|f\|_{L^\infty}, \|f\|_{L^\infty}\right]$. Then the case $f \neq g$ can be proved by noting that $f \cdot g = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$ and using p-Clarkson's inequality for $\widehat{\mathcal{E}}_p$ in (i).

(iv) This follows from an elementary fact on the Γ-convergence [Dal, Proposition 6.8].

(v) Let $f \in \mathcal{F}_p$ and let $\{f_k\}_k$ be a recovery sequence at f. It is clear that $f \circ T \in \mathcal{F}_p$. Note that $\|f \circ T - f_k \circ T\|_{L^p} = \|f - f_k\|_{L^p} \to 0$. Then

$$\mathcal{E}_p^\Gamma(f \circ T) \leq \liminf_{k\to\infty} \widetilde{\mathcal{E}}_p^{(n_k)}(f_k \circ T) = \liminf_{k\to\infty} \widetilde{\mathcal{E}}_p^{(n_k)}(f_k) \leq \mathcal{E}_p^\Gamma(f).$$

The converse $\mathcal{E}_p^\Gamma(f) \leq \mathcal{E}_p^\Gamma(f \circ T)$ can be shown by considering a recovery sequence at $f \circ T$. We complete the proof. □

Combining Proposition 6.21 and Theorem 6.22(iii), we can show the density of $\mathcal{F}_p \cap \mathcal{C}(K)$ in $\mathcal{C}(K)$. The density of $\mathcal{F}_p \cap \mathcal{C}(K)$ in $\mathcal{F}_p$ requires a long preparation and will be shown in Section 7.

Proof of Theorem 6.17(iv). By Proposition 6.21, $\mathcal{F}_p \cap \mathcal{C}(K)$ separates points of K. We note that, by Theorem 6.22(iii), $\mathcal{F}_p \cap \mathcal{C}(K)$ is a sub-algebra of $\mathcal{C}(K)$. So by the Stone–Weierstrass theorem, $\mathcal{F}_p \cap \mathcal{C}(K)$ is dense in $\mathcal{C}(K)$ with respect to the uniform norm. □

6.5 Poincaré type inequalities and partition of unity

In this subsection, we prove a Poincaré type inequality and provide a partition of unity with low energies.

Since we are yet to construct measures that play the role of "$|\nabla f|^p\,dm$", we express Poincaré inequality using re-scaled discrete p-energies. The following lemma allows us to obtain a version of Poincaré inequality from U-PI$_p(\beta)$.

Lemma 6.23. *There exists $C > 0$ (depending only on p and the doubling constant of m) such that the following holds: for any $x \in K$, $r > 0$ and $f \in L^p(K, m)$,*

$$\fint_{B_d(x,r)} \left|f(x) - f_{B_d(x,r)}\right|^p m(dx) \le C \liminf_{n\to\infty} \frac{1}{m\bigl(\widetilde{K}^{(n)}_{x,r}\bigr)} \sum_{w \in V_n(B_d(x,r))} \left|M_n f(w) - f_{\widetilde{K}^{(n)}_{x,r}}\right|^p m\bigl(\widetilde{K}_w\bigr),$$

where we set $\widetilde{K}^{(n)}_{x,r} = \bigcup_{w\in V_n(B_d(x,r))} \widetilde{K}_w$ ($n \in \mathbb{N}$) for ease of notation.

Proof. Let $x \in K$, $r > 0$ and $f \in L^p(K, m)$. For each $n \in \mathbb{N}$, define $f_n \coloneqq J_n(M_n f)$, where $J_n \colon \mathbb{R}^{V_n} \to L^0(K, m)$ is as given by (6.9). For all $n \in \mathbb{N}$ large enough so that $\widetilde{K}^{(n)}_{x,r} \subseteq B_d(x, 2r)$, we have

$$\begin{aligned}
&\frac{1}{m\bigl(\widetilde{K}^{(n)}_{x,r}\bigr)} \sum_{w \in V_n(B_d(x,r))} \left|M_n f(w) - f_{\widetilde{K}^{(n)}_{x,r}}\right|^p m\bigl(\widetilde{K}_w\bigr) \\
&\quad= \frac{1}{m\bigl(\widetilde{K}^{(n)}_{x,r}\bigr)} \sum_{w \in V_n(B_d(x,r))} \int_{\widetilde{K}_w} \left|f_n - f_{\widetilde{K}^{(n)}_{x,r}}\right|^p dm \\
&\quad\gtrsim \fint_{B_d(x,r)} \left|f_n - f_{\widetilde{K}^{(n)}_{x,r}}\right|^p dm \gtrsim \fint_{B_d(x,r)} \left|f_n - (f_n)_{B_d(x,r)}\right|^p dm,
\end{aligned}$$

where we used the volume doubling property of m and [BB, Lemma 4.17] in the last line. Since $\|f - f_n\|_{L^p} \to 0$ by the same argument as in Theorem 6.22, the dominated convergence theorem yields

$$\lim_{n\to\infty} \fint_{B_d(x,r)} \left|f_n - (f_n)_{B_d(x,r)}\right|^p dm = \fint_{B_d(x,r)} \left|f - f_{B_d(x,r)}\right|^p dm,$$

which proves our assertion. □

Now we prove a (p, p)-Poincaré-like inequality.

Lemma 6.24. *There exist $C > 0$ and $A \geq 1$ (depending only on the constants associated with Assumption 6.15) such that for all $x \in K$, $r > 0$ and $f \in L^p(K, m)$,*

$$\int_{B_d(x,r)} \left|f - f_{B_d(x,r)}\right|^p dm \leq Cr^{\beta} \liminf_{n\to\infty} \widetilde{\mathcal{E}}^{(n)}_{p,V_n(B_d(x,Ar))}(f). \tag{6.34}$$

Proof. Let $x \in K$, $r > 0$ and $f \in \mathcal{F}_p$. Let $\widetilde{K}^{(n)}_{x,r}$ be the same as in the previous lemma for each $n \in \mathbb{N}$. Let $C \geq 1$ be the constant in Definition 6.4 and choose $R_n > 0$ so that $R_n R_*^{-n} = 2Cr$. Note that $R_n \uparrow +\infty$ as $n \to \infty$. Since $\{\widetilde{K}_w\}_{w\in V_n}$ is a partition of K, there exists a unique $c_n \in V_n(B_d(x, r))$ such that $x \in \widetilde{K}_{c_n}$. For all $w \in V_n(B_d(x, r))$, by (6.5), (6.7), and picking a point $y \in B_d(x, r) \cap \widetilde{K}_v$,

$$\begin{aligned} d_n(c_n, w) \leq CR_*^n d(p_n(c_n), p_n(w)) &\leq CR_*^n\bigl(d(x, p_n(c_n)) + d(x, y) + d(y, p_n(v))\bigr) \\ &< CR_*^n\bigl(CR_*^{-n} + r + CR_*^{-n}\bigr) = 2C^2 + \frac{R_n}{2}. \end{aligned}$$

Hence we have $V_n(B_d(x, r)) \subseteq B_{d_n}(c_n, R_n)$ for all large enough $n \in \mathbb{N}$. By U-PI$_p(\beta)$, for all large $n \in \mathbb{N}$,

$$\begin{aligned} &\frac{1}{m\bigl(\widetilde{K}^{(n)}_{x,r}\bigr)} \sum_{w\in V_n(B_d(x,r))} \left|M_n f(w) - (M_n f)_{B_{d_n}(w_n,R_n)}\right|^p m\bigl(\widetilde{K}_w\bigr) \\ &\leq \frac{1}{m(B_d(x, r))} \sum_{v\in B_{d_n}(c_n,R_n)} \left|M_n f(w) - (M_n f)_{B_{d_n}(w_n,R_n)}\right|^p m\bigl(\widetilde{K}_w\bigr) \\ &\lesssim r^{-d_{\mathrm{f}}} R_*^{-nd_{\mathrm{f}}} \sum_{v\in B_{d_n}(c_n,R_n)} \left|M_n f(w) - (M_n f)_{B_{d_n}(w_n,R_n)}\right|^p \\ &\lesssim r^{-d_{\mathrm{f}}} R_*^{-nd_{\mathrm{f}}} R_n^{\beta} \mathcal{E}^{\mathbb{G}_n}_{p,B_{d_n}(c_n,A_{\mathrm{PI}}R_n)}(M_n f) \lesssim r^{-d_{\mathrm{f}}+\beta} R_*^{n(\beta-d_{\mathrm{f}})} \mathcal{E}^{\mathbb{G}_n}_{p,B_{d_n}(c_n,A_{\mathrm{PI}}R_n)}(M_n f). \end{aligned}$$

For any $v \in B_{d_n}(c_n, A_{\mathrm{PI}}R_n)$, by (6.5) and (6.7),

$$\widetilde{K}_v \subseteq B_d\bigl(x, 2CR_*^{-n} + CA_{\mathrm{PI}}R_n R_*^{-n}\bigr) \subseteq B_d\bigl(x, (2C^2 A_{\mathrm{PI}} + 1)r\bigr),$$

for all large $n \in \mathbb{N}$ so that $2CR_*^{-n} \leq r$. Let $A'_{\mathrm{PI}} \coloneqq 2C^2 A_{\mathrm{PI}} + 1$. Combining with [BB, Lemma 4.17], we obtain

$$\frac{1}{m\bigl(\widetilde{K}^{(n)}_{x,r}\bigr)} \sum_{w\in V_n(B_d(x,r))} \left|M_n f(w) - f_{\widetilde{K}^{(n)}_{x,r}}\right|^p m\bigl(\widetilde{K}_w\bigr) \lesssim r^{-d_{\mathrm{f}}+\beta} \widetilde{\mathcal{E}}^{(n)}_{p,V_n(B_d(x,A'_{\mathrm{PI}}r))}(f).$$

Letting $n \to \infty$ and using Lemma 6.23 completes the proof. □

We next provide a partition of unity with low energies without proofs. The following proposition is an immediate consequence of Theorem 6.22(i)-(iii). (See [MR, I-Exercise 4.16] in the case $p = 2$.)

Lemma 6.25. (i) *For any $f \in \mathcal{F}_p$ and $h \in \{|f|, f^+, f^-\}$, we have $\mathcal{E}_p^\Gamma(h) \le \mathcal{E}_p^\Gamma(f)$. Furthermore, there exists $C_p \ge 1$ depending only on p such that*

$$\mathcal{E}_p^\Gamma(f \wedge g) + \mathcal{E}_p^\Gamma(f \vee g) \le C_p\bigl(\mathcal{E}_p^\Gamma(f) + \mathcal{E}_p^\Gamma(g)\bigr) \quad \text{for all } f, g \in \mathcal{F}_p. \tag{6.35}$$

(ii) *Let $c, M > 0$ and let $f, g \in \mathcal{F}_p$ be non-negative functions such that $(f+g)|_{f\neq 0} \ge c$ and $f \le M$. Then there exists $D_{c,M}$ depending only on p, c, M such that*

$$\mathcal{E}_p^\Gamma\left(\frac{f}{f+g}\right) \le D_{c,M}\bigl(\mathcal{E}_p^\Gamma(f) + \mathcal{E}_p^\Gamma(g)\bigr). \tag{6.36}$$

Using Proposition 6.21, Lemma 6.25 and following a standard argument (see for instance, [Mur20, Lemma 2.5] in the case $p = 2$), we obtain a partition of unity with controlled energy.

Lemma 6.26. *Let $\varepsilon \in (0,1)$ and let V be a maximal ε-net of (K, d). Then there exists a family of functions $\{\psi_z\}_{z\in V}$ that satisfies the following properties:*

(i) $\sum_{z\in V} \psi_z \equiv 1$*;*

(ii) *For any $z \in V$, we have $\psi_z \in \mathcal{F}_p \cap \mathcal{C}(K)$ with $0 \le \psi_z \le 1$, $\psi_z\big|_{B_d(z,\varepsilon/4)} \equiv 1$ and* $\operatorname{supp}[\psi_z] \subseteq B_d(z, 5\varepsilon/4)$*;*

(iii) *If $z \in V$ and $z' \in V \setminus \{z\}$, then $\psi_{z'}\big|_{B_d(z,\varepsilon/4)} \equiv 0$.*

(iv) *There exists $C \ge 1$ (depending only on the constants associated with Assumption 6.15) such that $|\psi_z|_{\mathcal{F}_p}^p \le C\varepsilon^{d_{\mathrm{f}}-\beta}$ for all $z \in V$.*

7 Comparison with Korevaar–Schoen energies

In this section, we describe the Sobolev space $\mathcal{F}_p$ in terms of fractional Korevaar–Schoen energies. The associated function spaces are also called *Lipschitz–Besov spaces*. For Dirichlet forms on fractals endowed with nice heat kernel estimates, such characterizations are well-known; see, e.g., [GHL03, Jon96, Kum00, PP99].

In this section, we will always assume that the metric measure space (K, d, m) satisfies Assumption 6.15. For $r > 0$ and $A \in \mathcal{B}(K)$, define $J_{p,r}(\,\cdot\,; A)\colon L^p(K, m) \to [0, \infty)$ by

$$J_{p,r}(f; A) := \int_A \fint_{B_d(x,r)} |f(x) - f(y)|^p \, m(dy)m(dx) \quad \text{for each } f \in L^p(K, m).$$

We write $J_{p,r}(f)$ for $J_{p,r}(f; K)$ for ease of notation. The following main result in this section claims that our $(1, p)$-Sobolev space $\mathcal{F}_p$ coincides with the critical fractional Korevaar–Schoen space $B_{p,\infty}^{\beta/p}$ in this setting (recall Definition 1.3), where $\beta > 0$ is the exponent in Assumption 6.15.

Theorem 7.1. *Let (K,d,m) be a metric measure space satisfying Assumption 6.15. Then there exists $C \geq 1$ (depending only on the constants associated with Assumption 6.15) such that*

$$C^{-1}|f|_{\mathcal{F}_p}^p \leq \liminf_{r\downarrow 0} r^{-\beta} J_{p,r}(f) \leq \sup_{r>0} r^{-\beta} J_{p,r}(f) \leq C|f|_{\mathcal{F}_p}^p \quad \text{for any } f \in L^p(K,m). \tag{7.1}$$

In particular, $\mathcal{F}_p = B_{p,\infty}^{\beta/p}$ and $\sup_{r>0} r^{-\beta} J_{p,r}(f) \leq C^2 \underline{\lim}_{r\downarrow 0} r^{-\beta} J_{p,r}(f)$ for any $f \in L^p(K,m)$. Moreover, $\beta/p = s_p$, where s_p is the critical exponent defined in (1.3).

The proof of Theorem 7.1 will be divided into two parts. We start by showing $\sup_{r>0} r^{-\beta} J_{p,r}(f) \lesssim |f|_{\mathcal{F}_p}$ using the Poincaré inequality in Lemma 6.24.

Lemma 7.2. *There exists $C > 0$ (depending only on the constants associated with Assumption 6.15) such that for all Borel set U of K and $f \in L^p(K,m)$,*

$$\limsup_{r\downarrow 0} r^{-\beta} J_{p,r}(f;U) \leq C \limsup_{r\downarrow 0} \liminf_{n\to\infty} \widetilde{\mathcal{E}}_{p,V_n(U_r)}^{(n)}(f), \tag{7.2}$$

where U_r denotes the r-neighborhood of U, i.e., $U_r = \bigcup_{y\in U} B_d(y,r)$ for each $r > 0$. Moreover, it holds that $\sup_{r>0} r^{-\beta} J_{p,r}(f) \leq C|f|_{\mathcal{F}_p}^p$.

Proof. Let $r > 0$ and let $N_r \subseteq U$ be a maximal r-net of U with respect to the metric d. Note that $B_d(x,r) \subseteq B_d(y,2r)$ for $y \in N_r$ and $x \in B_d(y,r)$. We see that

$$\begin{aligned}
r^{-\beta} J_{p,r}(f;U) &\leq \sum_{y\in N_r} r^{-\beta} J_{p,r}(f; B_d(y,r)) \\
&\lesssim \sum_{y\in N_r} \int_{B_d(y,2r)} \fint_{B_d(y,2r)} \frac{|f(x)-f(y)|^p}{r^\beta}\, m(dy)m(dx) \qquad \text{(by VD)} \\
&\lesssim \sum_{y\in N_r} \int_{B_d(y,2r)} \fint_{B_d(y,2r)} \left\{ \frac{\left|f(x)-f_{B_d(y,2r)}\right|^p}{r^\beta} + \frac{\left|f(y)-f_{B_d(y,2r)}\right|^p}{r^\beta} \right\} m(dy)m(dx) \\
&\lesssim \sum_{y\in N_r} \liminf_{n\to\infty} \widetilde{\mathcal{E}}_{p,V_n(B_d(y,2Ar))}^{(n)}(f). \qquad \text{(by Lemma 6.24)}
\end{aligned} \tag{7.3}$$

For any $y \in N_r$ and $w \in V_n(B_d(y,2Ar))$, it is immediate that $w \in V_n(U_{2Ar})$. The overlap of $\left\{V_n(B_d(y,2Ar))\right\}_{y\in N_r}$ can be controlled in the following manner. Let $y \in N_r$ and let $n \in \mathbb{N}$ be large enough so that $CR_*^{-n} < r$, where $C \geq 1$ is the constant in Definition 6.4. Then we easily see that $\{p_n(w)\}_{w\in V_n(B_d(y,2Ar))} \subseteq B_d(y,(2A+1)r)$. In particular, we have

$$\max_{w\in V_n} \#\left\{y \in N_r \,\middle|\, w \in V_n(B_d(y,2Ar))\right\} \leq \sup_{x\in K} \#\left\{y \in N_r \,\middle|\, x \in B_d(y,(2A+1)r)\right\} \lesssim 1, \tag{7.4}$$

where we used the metric doubling property in the last inequality.

Let us go back to the estimate on $\sum_{y\in N_r}\varliminf_{n\to\infty}\widetilde{\mathcal{E}}^{(n)}_{p,V_n(B_d(y,2Ar))}(f)$. By (7.4),

$$\sum_{y\in N_r}\varliminf_{n\to\infty}\widetilde{\mathcal{E}}^{(n)}_{p,V_n(B_d(y,2Ar))}(f)\le\varliminf_{n\to\infty}\sum_{y\in N_r}\widetilde{\mathcal{E}}^{(n)}_{p,V_n(B_d(y,2Ar))}(f)\lesssim\varliminf_{n\to\infty}\widetilde{\mathcal{E}}^{(n)}_{p,V_n(U_{2Ar})}(f). \tag{7.5}$$

Combining with (7.3) and taking the limsup as $r\downarrow 0$ proves (7.2).

In the case $U=K$, by considering $|f|^p_{\mathcal{F}_p}$ instead of $\varliminf_{n\to\infty}\widetilde{\mathcal{E}}^{(n)}_{p,V_n(U_{2Ar})}(f)$ in (7.5), we immediately get $r^{-\beta}J_{p,r}(f)\lesssim|f|^p_{\mathcal{F}_p}$, which completes the proof. □

Next we move to the converse bound: $\liminf_{r\downarrow 0}r^{-\beta}J_{p,r}(f)\gtrsim|f|^p_{\mathcal{F}_p}$. Our approach is similar to [Bau24, Theorem 5.2] but we give a local version as well.

Lemma 7.3. *There exists $C>0$ (depending only on the constants associated with Assumption 6.15) such that the following hold. For all $U\subseteq K$ and $f\in\mathcal{F}_p$,*

$$\limsup_{n\to\infty}\widetilde{\mathcal{E}}^{(n)}_{p,V_n(U)}(f)\le C\lim_{\delta\downarrow 0}\liminf_{r\downarrow 0}r^{-\beta}J_{p,r}(f;U_\delta), \tag{7.6}$$

where U_δ denotes the δ-neighborhood of U. Furthermore, for any $f\in L^p(K,m)$,

$$|f|^p_{\mathcal{F}_p}\le C\liminf_{r\downarrow 0}r^{-\beta}J_{p,r}(f). \tag{7.7}$$

Proof. Let $r\in(0,1)$ and fix a maximal r-net $N_r(U)\subseteq U$ of U. Let N_r be a maximal r-net of (K,d) such that $N_r(U)\subseteq N_r$. We first observe that, by (6.5) and (6.7), there exists $\varepsilon>0$ such that for all $n\in\mathbb{N}$ satisfying $R_*^{-n}<\varepsilon r$, we have

$$\widetilde{K}_v\cup\widetilde{K}_w\subseteq B_d(z,5r/4)\quad\text{whenever } z\in K,\ \{v,w\}\in E_n \text{ and } v\in V_n(B_d(z,r))\ .$$

Therefore, for all large $n\in\mathbb{N}$ and $f\in L^p(K,m)$,

$$\widetilde{\mathcal{E}}^{(n)}_{p,V_n(U)}(f)\le\sum_{z\in N_r(U)}\widetilde{\mathcal{E}}^{(n)}_{p,V_n(B_d(z,5r/4))}(f).$$

Let $\{\psi_{z,r}\}_{z\in N_r}$ satisfy the conditions (i)-(iv) in Lemma 6.26. To estimate $\widetilde{\mathcal{E}}^{(n)}_{p,V_n(B_d(z,5r/4))}(f)$, we introduce a linear operator $A_r\colon L^p(K,m)\to L^p(K,m)$ as

$$A_rf:=\sum_{z\in N_r}f_{B_d(z,r/4)}\psi_{z,r},\quad f\in L^p(K,m).$$

Let us observe a few properties of A_r. Note that $A_rf\in\mathcal{F}_p\cap\mathcal{C}(K)$. By Hölder's inequality, $\sum_{z\in N_r}\psi_{z,r}\le 1$ and the doubling property, for any $f\in L^p(K,m)$,

$$\begin{aligned}\|A_rf\|^p_{L^p}&\le\int_K\left(\sum_{z\in N_r}\left|f_{B_d(z,r/4)}\right|^p\psi_{z,r}(x)\right)\left(\sum_{z\in N_r}\psi_{z,r}(x)\right)^{p-1}m(dx)\\&\le\int_K\left(\sum_{z\in N_r}\frac{1}{m\bigl(B_d(z,r/4)\bigr)}\int_{B_d(z,r/4)}|f|^p\,dm\right)\mathbb{1}_{B_d(z,r)}(x)\,m(dx)\\&=\sum_{z\in N_r}\frac{m\bigl(B_d(z,r)\bigr)}{m\bigl(B_d(z,r/4)\bigr)}\int_{B_d(z,r/4)}|f|^p\,dm\lesssim\|f\|^p_{L^p}\,,\end{aligned}$$

whence it follows that $\|A_r f\|_{L^p\to L^p} \le C_0$ for some constant $C_0 > 0$ that is independent of r. For $g \in \mathcal{C}(K)$, we easily show that $A_r g \to g$ in the uniform norm as $r \downarrow 0$ by virtue of the uniform continuity of g and the properties of $\{\psi_{z,r}\}_{z\in N_r}$. Now we can show that $\|f - A_r f\|_{L^p} \to 0$ as $r \downarrow 0$ for any $f \in L^p(K, m)$. Let $\varepsilon > 0$ and choose $g_\varepsilon \in \mathcal{C}(K)$ so that $\|f - g_\varepsilon\|_{L^p} < \varepsilon$. Then we have

$$\begin{aligned}\|f - A_r f\|_{L^p} &\le \|f - g_\varepsilon\|_{L^p} + \|g_\varepsilon - A_r g_\varepsilon\|_{L^p} + \|A_r g_\varepsilon - A_r f\|_{L^p}\\ &\le \varepsilon + \|g_\varepsilon - A_r g_\varepsilon\|_{L^p} + C_0\varepsilon,\end{aligned}$$

and hence $\limsup_{r\downarrow 0}\|f - A_r f\|_{L^p} \le (1 + C_0)\varepsilon$, proving $\|f - A_r f\|_{L^p} \to 0$.

For $z \in N_r$ and $x \in B_d(z, 3r/2)$, we observe

$$A_r f(x) = f_{B_d(z,r/4)} + \sum_{w\in N_r\cap B_d(z,11r/4)} \bigl(f_{B_d(w,r/4)} - f_{B_d(z,r/4)}\bigr)\psi_{w,r}(x).$$

Since N_r is a r-net, there exists $M \in \mathbb{N}$ depending only on the doubling constant such that $\#\bigl(N_r\cap B_d(w, 11r/4)\bigr) \le M$ for any $w \in N_r$. Also, since $\bigcup_{w\in V_n(B_d(z,5r/4))} \widetilde{K}_w \subseteq B_d(z, 3r/2)$ for all large $n \in \mathbb{N}$, we see that

$$M_n(A_r f) = f_{B_d(z,r/4)} + \sum_{w\in N_r\cap B_d(z,11r/4)} \bigl(f_{B_d(w,r/4)} - f_{B_d(z,r/4)}\bigr) M_n\psi_{w,r} \quad \text{on } V_n\bigl(B_d(z, 5r/4)\bigr).$$

Hence we have

$$\begin{aligned}&\widetilde{\mathcal{E}}^{(n)}_{p,V_n(B_d(z,5r/4))}(A_r f)\\ &= R_*^{n(\beta-d_{\mathrm{f}})}\mathcal{E}^{\mathbb{G}_n}_{p,V_n(B_d(z,5r/4))}\left(\sum_{w\in N_r\cap B_d(z,11r/4)} \bigl(f_{B_d(w,r/4)} - f_{B_d(z,r/4)}\bigr) M_n\psi_{w,r}\right)\\ &\le M^{p-1}\sum_{w\in N_r\cap B_d(z,11r/4)} \bigl|f_{B_d(w,r/4)} - f_{B_d(z,r/4)}\bigr|^p \widetilde{\mathcal{E}}^{(n)}_{p,V_n(B_d(z,5r/4))}(\psi_{w,r})\\ &\lesssim r^{d_{\mathrm{f}}-\beta}\sum_{w\in N_r\cap B_d(z,11r/4)} \bigl|f_{B_d(w,r/4)} - f_{B_d(z,r/4)}\bigr|^p. \end{aligned} \tag{7.8}$$

For $z, w \in N_r$ with $w \in B_d(z, 11r/4)$, by Hölder's inequality and the Ahlfors regularity of m, we easily obtain

$$r^{d_{\mathrm{f}}}\bigl|f_{B_d(w,r/4)} - f_{B_d(z,r/4)}\bigr|^p \lesssim \int_{B_d(w,3r)} \fint_{B_d(x,9r)} |f(x) - f(y)|^p\, m(dy)m(dx),$$

which together with (7.8) yields

$$\begin{aligned}&\widetilde{\mathcal{E}}^{(n)}_{p,V_n(B_d(z,5r/4))}(A_r f)\\ &\lesssim r^{-\beta}\sum_{w\in N_r\cap B_d(z,11r/4)} \int_{B_d(w,3r)} \fint_{B_d(x,9r)} |f(x) - f(y)|^p\, m(dy)m(dx).\end{aligned} \tag{7.9}$$

Let us fix $\delta > 0$. Then, for all small enough $r > 0$ and $z \in N_r(U)$, we have $\bigcup_{w \in N_r \cap B_d(z,11r/4)} B_d(w,3r) \subseteq U_\delta$. Summing (7.9) over $z \in N_r(U)$, we obtain

$$\begin{aligned}\widetilde{\mathcal{E}}^{(n)}_{p,V_n(U)}(A_r f) &\le \sum_{z \in N_r(U)} \widetilde{\mathcal{E}}^{(n)}_{p,V_n(B_d(z,5r/4))}(A_r f) \\ &\lesssim r^{-\beta} \int_{U_\delta} \fint_{B_d(x,9r)} |f(x) - f(y)|^p \, m(dy) m(dx) \lesssim (9r)^{-\beta} J_{p,9r}(f; U_\delta), \quad (7.10)\end{aligned}$$

where we used the metric doubling property in order to control the overlap of $\{B_d(w,3r) \mid w \in N_r \cap B_d(z,11r/4)\}$ in the second inequality. Note that (7.10) holds for large enough $n \in \mathbb{N}$ so that $R_*^{-n} < \varepsilon r$, where $\varepsilon > 0$ is as given in the beginning of the proof.

To show (7.6), we may and shall assume that $\liminf_{r \downarrow 0} r^{-\beta} J_{p,r}(f; U_\delta) < \infty$. Pick a sequence $\{r_k\}_{k \in \mathbb{N}}$ such that $r_k \downarrow 0$ as $k \to \infty$ and $\lim_{k \to \infty} r_k^{-\beta} J_{p,r_k}(f; U_\delta) = \varliminf_{r \downarrow 0} r^{-\beta} J_{p,r}(f; U_\delta)$. If $f \in \mathcal{F}_p$, then (7.10) with $U = K$ and Lemma 7.2 tell us that $\left|A_{r_k/9} f\right|^p_{\mathcal{F}_p} \lesssim r_k^{-\beta} J_{p,r_k}(f) \lesssim |f|^p_{\mathcal{F}_p} < \infty$. In particular, $\{A_{r_k/9} f\}_{k \in \mathbb{N}}$ is bounded in $\mathcal{F}_p$. Hence, by taking a subsequence, we can assume that $f_k := A_{r_k/9} f$ converges weakly in $\mathcal{F}_p$ to some function $f_\infty \in \mathcal{F}_p$. Since $\mathcal{F}_p$ is continuously embedded in $L^p(K,m)$, we have $f_\infty = f$. By Mazur's lemma (Lemma A.1) and (7.10), we obtain (7.6).

By setting $U = K$ in (7.6) and using (6.22), we obtain (7.7). □

We now prove the main result (Theorem 7.1) of this section.

Proof of Theorem 7.1. The desired comparability follows from Lemmas 7.2 and 7.3, so we show $\beta/p = s_p$. Since $\mathcal{F}_p = B^{\beta/p}_{p,\infty}$, the bound $\beta/p \le s_p$ is immediate from Theorem 6.17(iv). To prove the converse, let $s > \beta/p$ and let $f \in \mathcal{F}_p \supseteq B^s_{p,\infty}$ such that $|f|_{\mathcal{F}_p} > 0$, i.e. $f \in \mathcal{F}_p \setminus \mathbb{R}\mathbb{1}_K$. Let $\mathcal{A}_n := A_{R_*^{-n}/9}$, where A_r $(r > 0)$ is the same operator as in the proof of Lemma 7.3. Then, by (7.10) with $r = R_*^{-n}/9$ for large enough $n \in \mathbb{N}$ and Theorem 6.22, we have $R_*^{-n(\beta - sp)} \mathcal{E}^\Gamma_p(\mathcal{A}_n f) \lesssim R_*^{nsp} J_{p,R_*^{-n}}(f)$. Since $\varliminf_{n \to \infty} \mathcal{E}^\Gamma_p(\mathcal{A}_n f) \gtrsim |f|^p_{\mathcal{F}_p} > 0$, by combining with $-(\beta - sp) > 0$, we conclude that $\varliminf_{r \downarrow 0} r^{-sp} J_{p,r}(f) = \infty$ for any $f \in \mathcal{F}_p \setminus \mathbb{R}\mathbb{1}_K$, which proves $B^s_{p,\infty} = \mathbb{R}\mathbb{1}_K$. This completes the proof. □

Finally, we can prove the density of $\mathcal{F}_p \cap \mathcal{C}(K)$ in $\mathcal{F}_p$.

Proof of Theorem 6.17(v). Set $\widehat{\mathcal{F}}_p := \overline{\mathcal{F}_p \cap \mathcal{C}(K)}^{\|\cdot\|_{\mathcal{F}_p}}$. The inclusion $\widehat{\mathcal{F}}_p \subseteq \mathcal{F}_p$ is obvious. So, it suffices to prove $\mathcal{F}_p \subseteq \widehat{\mathcal{F}}_p$.

By Theorem 7.1, we know that $\mathcal{F}_p = B^{\beta/p}_{p,\infty}$. Let $f \in \mathcal{F}_p$ and let A_r $(r > 0)$ be the operators defined in the proof of Lemma 7.3. Then $A_r f \in \mathcal{F}_p \cap \mathcal{C}(K) \subseteq \widehat{\mathcal{F}}_p$. By (7.10) with $U = K$, we have $|A_r f|^p_{\mathcal{F}_p} \lesssim \sup_{r>0} r^{-\beta} J_{p,r}(f) \lesssim |f|^p_{\mathcal{F}_p} < \infty$. Combining with $\|A_r f\|_{L^p} \lesssim \|f\|_{L^p}$, we conclude that $\{A_r f\}_{r>0}$ is bounded in $\mathcal{F}_p$. Let $\{A_{r_k} f\}_{k \in \mathbb{N}}$ be a convergent subsequence of $\{A_r f\}_{r>0}$ with respect to the weak topology of $\mathcal{F}_p$. Applying Mazur's lemma (Lemma A.1), we obtain

$$f \in \overline{\{\text{convex combinations of } \{A_{r_k} f\}_{k\in\mathbb{N}}\}}^{\|\cdot\|_{\mathcal{F}_p}} \subseteq \overline{\mathcal{F}_p \cap \mathcal{C}(K)}^{\|\cdot\|_{\mathcal{F}_p}} = \widehat{\mathcal{F}}_p,$$

which completes the proof of Theorem 6.17. □

8 Sobolev spaces on the Sierpiński carpet

The aim of this section is to prove the first four main results in the introduction: Theorems 1.1, 1.4 1.2 and 1.5. Some results (e.g. constructions of self-similar energies and energy measures under suitable hypotheses) can be extended to a general setting, but we focus on the case of the *planar standard Sierpiński carpet* for the sake of simplicity and refer to [MS] for such generalizations.

First, recall the definition of the Sierpiński carpet and basic notions on self-similar sets (see [Kig01, Chapter 1] for further background).

Definition 8.1 (Planar Sierpiński carpet)**.** (1) Let $a_* = 3, N_* = 8, S = \{1, \dots, N_*\}$ and define $q_i \in \mathbb{R}^2, i \in S$ as

$$q_1 = (-1,-1) = -q_5, \quad q_2 = (0,-1) = -q_6,$$
$$q_3 = (1,-1) = -q_7, \quad q_4 = (1,0) = -q_8.$$

Let $f_i \colon \mathbb{R}^2 \to \mathbb{R}^2, i \in S$ denote the similitude $f_i(x) = a_*^{-1}(x - q_i) + q_i$. Let K be the unique non-empty compact subset such that $K = \bigcup_{i\in S} f_i(K)$ and set $F_i = f_i\big|_K$. Let d denote the normalized Euclidean metric on K so that $\mathrm{diam}(K,d) = 1$. The tuple $(K, S, \{F_i\}_{i\in S})$ is called the *planar standard Sierpiński carpet* (PSC for short). Let m be the self-similar probability measure with uniform weight, that is, $m = N_*^{-1}\sum_{i\in S} m \circ F_i^{-1}$.

(2) Let $\ell_{\mathrm{L}} := \{-1\} \times [-1,1]$, $\ell_{\mathrm{T}} := [-1,1] \times \{1\}$, $\ell_{\mathrm{R}} := \{1\} \times [-1,1]$ and $\ell_{\mathrm{B}} := [-1,1] \times \{-1\}$. Define $\mathcal{V}_0 := \partial[-1,1]^2 = \ell_{\mathrm{L}} \cup \ell_{\mathrm{T}} \cup \ell_{\mathrm{R}} \cup \ell_{\mathrm{B}}$.

(3) Let D_4 be the dihedral group of order 8 (the symmetry of the square), i.e.

$$D_4 = \{R_k, S_k \mid k = 0,1,2,3\},$$

where

$$R_k = \begin{bmatrix} \cos\frac{k\pi}{2} & -\sin\frac{k\pi}{2} \\ \sin\frac{k\pi}{2} & \cos\frac{k\pi}{2} \end{bmatrix} \quad \text{and} \quad S_k = \begin{bmatrix} \cos\frac{k\pi}{2} & \sin\frac{k\pi}{2} \\ \sin\frac{k\pi}{2} & -\cos\frac{k\pi}{2} \end{bmatrix}.$$

Then it is clear that $\Phi(K) = K$ for all $\Phi \in D_4$.

Definition 8.2 (Words and shift space)**.** Let S, $\{F_i\}_{i\in S}$ be as given in Definition 8.1. For convention, we set $S^0 \coloneqq \{\phi\}$, where ϕ is an element called the *empty word*. Let $W_n \coloneqq S^n$ for each $n \in \mathbb{Z}_{\geq 0}$ and define $W_* \coloneqq \bigcup_{n\geq 0} W_n$. For $w = w_1 w_2 \cdots w_n \in W_n$, define $F_w \coloneqq F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_n}$, $K_w \coloneqq F_w(K)$ and $|w| \coloneqq n$. Let Σ be the *one-sided shift space* of symbols S, that is, $\Sigma = \{\omega = \omega_1\omega_2\omega_3 \cdots \mid \omega_i \in S \text{ for any } i \in \mathbb{N}\}$. We endow Σ with the product topology which makes it a compact metrizable space [Kig01, Theorem 1.2.2]. Define the *shift map* $\sigma\colon \Sigma \to \Sigma$ by $\sigma(\omega_1\omega_2\cdots) = \omega_2\omega_3\cdots$ for each $\omega_1\omega_2\cdots \in \Sigma$. The branches of σ are denoted by σ_i $(i \in S)$, i.e. $\sigma_i\colon \Sigma \to \Sigma$ is defined as $\sigma_i(\omega_1\omega_2\cdots) = i\omega_1\omega_2\cdots$ for each $i \in S$ and $\omega_1\omega_2\cdots \in \Sigma$. For $\omega = \omega_1\omega_2\cdots \in \Sigma$ and $k \in \mathbb{Z}_{\geq 0}$, we define $[\omega]_k = \omega_1\cdots\omega_k \in S^k$. Similarly, define $[w]_k = w_1\cdots w_k \in W_k$ for any $w = w_1\cdots w_n \in W_n$ with $n \geq k$. For $w \in W_n$ and $m \in \mathbb{N}$, let $\Sigma_w \coloneqq \sigma_w(\Sigma) = \{\omega \in \Sigma \mid [\omega]_n = w\}$ and $S^m(w) \coloneqq \{v \in W_{n+m} \mid [v]_n = w\}$. We use $S(w)$ to denote $S^1(w)$ for simplicity. Let $\chi\colon \Sigma \to K$ be the continuous surjection satisfying $\{\chi(\omega)\} = \bigcap_{n\geq 0} K_{[\omega]_n}$ for all $\omega \in \Sigma$ [Kig01, Proposition 1.3.3].

Hereafter, we let $(K, S, \{F_i\}_{i\in S})$ be PSC, d be the normalized metric, and m be the self-similar probability measure on K as given in Definition 8.1. Note that $K_v \cap K_w = F_v(\mathcal{V}_0) \cap F_w(\mathcal{V}_0)$ for any $v \neq w \in W_*$ with $|v| = |w|$ [Kig01, Proposition 1.3.5(2)]. Let us fix a family of Borel sets $\{\widetilde{K}_w\}_{w\in W_*}$ satisfying $\mathrm{int}_K K_w \subseteq \widetilde{K}_w \subseteq K_w$ for any $w \in W_*$ and $\widetilde{K}_v \cap \widetilde{K}_w = \emptyset$ for any $v \neq w \in W_*$ with $|v| = |w|$. Consider the approximating graphs $\{\mathbb{G}_n = (V_n, E_n)\}_{n\in\mathbb{N}}$ given by

$$V_n \coloneqq W_n = S^n, \quad E_n \coloneqq \bigl\{\{v, w\} \in V_n \times V_n \bigm| v \neq w, K_v \cap K_w \neq \emptyset\bigr\},$$

and that $M_n\colon L^p(K, m) \to \mathbb{R}^{V_n}$ in (6.8) is defined as

$$M_n f(w) = \fint_{\widetilde{K}_w} f\,dm \quad \text{for } f \in L^p(K, m) \text{ and } w \in W_n.$$

For $A \subseteq K$ and $n \in \mathbb{Z}_{\geq 0}$, define $W_n[A] \coloneqq \{w \in W_n \mid K_w \cap A \neq \emptyset\}$.

8.1 Construction of a self-similar energy: Theorems 1.1 and 1.4

To carry out the strategy in Section 6, a crucial step is to verify Assumption 6.15 for PSC. Besides, we have to check the *pre-self-similarity condition* (see Theorem 8.3(c) below) in order to get a self-similar p-energy by applying a known result of Kigami [Kig00]. So the following theorem is an important preparation whose proof is divided into several steps.

Theorem 8.3. *PSC satisfies Assumption 6.15 for all* $p \in (1, \infty)$*, that is,*

(a) (K, d, m) *is* d_{f}*-Ahlfors regular, where* $d_{\mathrm{f}} \coloneqq \log N_*/\log a_* = \log 8/\log 3$*. In addition, the sequence of graphs* $\{\mathbb{G}_n = (V_n, E_n)\}_{n\in\mathbb{N}}$ *equipped with the projective map* $\pi_{n,k}$ $(1 \leq k < n)$*, which is defined as* $\pi_{n,k}(w) \coloneqq [w]_k$ $(w \in V_n)$*, is* a_**-scaled and* a_**-compatible with* (K, d)*.*

(b) *The sequence* $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ *satisfies* U-PI$_p(d_{\mathrm{w}}(p))$ *and* U-CF$_p(\vartheta, d_{\mathrm{w}}(p))$ *for some* $\vartheta \in (0,1]$*, where* $d_{\mathrm{w}}(p) = \log\bigl(N_*\rho(p)\bigr)/\log a_*$ *and* $\rho(p) \in (0,\infty)$ *is given later (see* (8.5)*).*

Moreover, the following pre-self-similarity condition holds:

(c) $f \circ F_i \in \mathcal{F}_p$ *for all* $i \in S$ *and* $f \in \mathcal{F}_p$*. Furthermore,*

$$\mathcal{F}_p \cap \mathcal{C}(K) = \{f \in \mathcal{C}(K) \mid f \circ F_i \in \mathcal{F}_p \text{ for all } i \in S\}, \tag{8.1}$$

and the semi-norm $|f|_{\mathcal{F}_p} = \Bigl(\sup_{n\in\mathbb{N}} a_*^{n(d_{\mathrm{w}}(p)-d_{\mathrm{f}})}\mathcal{E}_p^{\mathbb{G}_n}(M_n f)\Bigr)^{1/p}$ *satisfies the following: there exists* $C \geq 1$ *such that for all* $n \in \mathbb{N}$ *and* $f \in \mathcal{F}_p$*,*

$$C^{-1}|f|_{\mathcal{F}_p}^p \leq \rho(p)^n \sum_{w\in W_n} |f \circ F_w|_{\mathcal{F}_p}^p \leq C|f|_{\mathcal{F}_p}^p. \tag{8.2}$$

We note that the estimate (8.2) follows from properties (iii) and (viii) in Theorem 1.1. Therefore (8.2) is necessary for the conclusion of Theorem 1.1.

We start by observing the geometry of PSC, namely Theorem 8.3(a). The next proposition gives a collection of geometric properties of PSC.

Proposition 8.4. (i) *For all* $n \in \mathbb{Z}_{\geq 0}$ *and distinct* $v, w \in W_n$*, we have* $m(K_w) = N_*^{-n}$ *and* $m(K_v \cap K_w) = 0$*.*

(ii) *There exists* $C \geq 1$ *(depending only on* a_**) such that the following hold: for all* $n \in \mathbb{Z}_{\geq 0}$ *and* $w \in W_n$*, there exists* $x \in K_w$ *satisfying*

$$B_d(x, C^{-1}a_*^{-n}) \subseteq K_w \subseteq B_d(x, Ca_*^{-n}).$$

In particular, (6.7) *holds.*

(iii) *There exists* C_{AR} *depending only on* a_* *and* N_* *such that*

$$C_{\mathrm{AR}}^{-1} r^{d_{\mathrm{f}}} \leq m(B_d(x,r)) \leq C_{\mathrm{AR}}\, r^{d_{\mathrm{f}}} \quad \text{for all } x \in K,\ r \in (0,1],$$

i.e., (K, d, m) *is* d_{f}*-Ahlfors regular.*

(iv) $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ *equipped with the projective maps* $\{\pi_{n,k} \mid n,k \in \mathbb{N}, k < n\}$ *is* a_**-scaled.*

(v) $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ *equipped with the projective maps* $\{\pi_{n,k} \mid n,k \in \mathbb{N}, k < n\}$ *is* a_**-compatible.*

(vi) *For any* $\Phi \in D_4$*, there exists a bijection* $\tau_\Phi \colon W_* \to W_*$ *such that* $|\tau_\Phi(w)| = |w|$ *and* $\Phi\bigl(K_w\bigr) = K_{\tau_\Phi(w)}$ *for any* $w \in W_*$*. Moreover,* $U_{\Phi,w} := F_{\tau_\Phi(w)}^{-1} \circ \Phi \circ F_w \in D_4$*.*

In particular, Theorem 8.3(a) *holds.*

Proof. The properties (ii), (vi) are easy and (iii) is a consequence of (i), (ii). So we will prove (i), (iv) and (v).

(i) This follows from $\overline{\mathcal{V}_0} \neq K$ and [Kig09, Theorem 1.2.7].

(iv) Recall that d_n denotes the graph distance of $\mathbb{G}_n$. Let $n, m \in \mathbb{N}$ and $w \in W_m$. Let $c_n(w) = w15^{n-1} \in V_{n+m}$. Then it is clear that $B_{d_{n+m}}\bigl(c_n(w), a_*^{n-1}\bigr) \subseteq \pi_{n+m,m}^{-1}(w)$. Since we can easily see that $\operatorname{diam}\bigl(\pi_{n+m,m}^{-1}(w), d_{n+m}\bigr) \leq 2a_*^n$, we obtain $\pi_{n+m,m}^{-1}(w) \subseteq B_{d_{n+m}}\bigl(c_n(w), 3a_*^n\bigr)$. Hence we have (6.3) with $A_1 = 3 \vee a_*$. Also, the bound on the diameter of $\pi_{n+m,m}^{-1}(\,\cdot\,)$ implies (6.4) with $A_2 = 4$. This completes the proof.

(v) Note that the conditions in Definition 6.4(ii),(iii) are already verified. Let $p_n(v) = F_v\bigl(F_1(1,1)\bigr) \in K_v$ for $n \in \mathbb{N}$ and $v \in V_n$. Then the condition in Definition 6.4(iv) is evident. So we will prove the Hölder comparison (6.5). Let $v, w \in V_n$ with $v \neq w$. Pick a path $[z(0), \dots, z(l)]$ in $\mathbb{G}_n$ such that $\{z(0), z(l)\} = \{v, w\}$ and $l \leq d_n(v, w)$. Then

$$d(p_n(v), p_n(w)) \leq \operatorname{diam}\left(\bigcup_{j=0}^{l} K_{z(j)}, d\right) \leq 2la_*^{-n},$$

which implies the upper bound in (6.5) with $C = 2$.

The desired lower bound requires a geometric observation. Let $\pi_i \colon \mathbb{R}^2 \to \mathbb{R}\,(i = 1, 2)$ denote the projection map of $\mathbb{R}^2$ onto i-th coordinate, i.e. $\pi_i(x_1, x_2) = x_i$ for $(x_1, x_2) \in \mathbb{R}^2$. Then we observe that

$$|\pi_1(p_n(v)) - \pi_1(p_n(w))| \vee |\pi_2(p_n(v)) - \pi_2(p_n(w))| \geq \frac{d_n(v, w)}{2} \cdot 2a_*^{-n-1},$$

which implies $d(p_n(v), p_n(w)) \geq (2\sqrt{2}a_*)^{-1} d_n(v, w) a_*^{-n}$. Therefore, (6.5) holds with $C = 2\sqrt{2}a_*$. □

We next move to Theorem 8.3(b). Thanks to Propositions 6.8 and 6.12, checking U-cap$_{p,\leq}(d_{\mathrm{w}}(p))$ and U-BCL$_p^{\mathrm{low}}(d_{\mathrm{f}} - d_{\mathrm{w}}(p))$ is enough for this purpose. The planarity is crucial to ensure $d_{\mathrm{f}} - d_{\mathrm{w}}(p) < 1$ for **any** $p \in (1, \infty)$. We start with the definition of $d_{\mathrm{w}}(p)$ which is the quantity called the *p-walk dimension* of PSC (see Definition 8.7 below). This value is closely related with the sub-multiplicative and super-multiplicative inequalities of discrete p-capacities due to Bourdon and Kleiner.

Theorem 8.5 ([BK13, Lemma 4.4])**.** *Let $p \in [1, \infty)$. Define*

$$\mathcal{C}_p^{(n)} := \sup_{m \in \mathbb{N}, w \in V_m} \operatorname{cap}_p^{\mathbb{G}_{n+m}}\bigl(\pi_{n+m,m}^{-1}(w), V_{n+m} \setminus \pi_{n+m,m}^{-1}(B_{d_m}(w, 2))\bigr). \tag{8.3}$$

Then there exists $C \geq 1$ (depending only on p, L_*) such that

$$C^{-1} \cdot \mathcal{C}_p^{(n)} \mathcal{C}_p^{(m)} \leq \mathcal{C}_p^{(n+m)} \leq C \cdot \mathcal{C}_p^{(n)} \mathcal{C}_p^{(m)} \quad \text{for all } n, m \in \mathbb{N}. \tag{8.4}$$

In particular, the limit

$$\lim_{n \to \infty} \bigl(\mathcal{C}_p^{(n)}\bigr)^{-1/n} =: \rho(p) \in (0, \infty) \tag{8.5}$$

exists, and

$$C^{-1}\rho(p)^{-n} \le \mathcal{C}_p^{(n)} \le C\rho(p)^{-n} \quad \textit{for all } n \in \mathbb{N}. \tag{8.6}$$

We call $\rho(p)$ *the* p-scaling factor *of PSC.*

Remark 8.6. The work [BK13] uses a slightly different version of $\mathcal{C}_p^{(n)}$. The value of M_n in [BK13, Lemma 4.4] is uniformly comparable with $\mathcal{C}_p^{(n)}$ (cf. Lemma 2.12 and [BK13, last line in p. 66]).

Definition 8.7. Let $p \ge 1$. Define

$$d_{\mathrm{w}}(p) \coloneqq \frac{\log\left(N_*\rho(p)\right)}{\log a_*}. \tag{8.7}$$

We call $d_{\mathrm{w}}(p)$ the *p-walk dimension* of PSC.

The next proposition is a collection of properties concerning analytic conditions.

Proposition 8.8. (i) $d_{\mathrm{f}} - d_{\mathrm{w}}(p) < 1$ *for all* $p \in [1,\infty)$.

(ii) *The sequence* $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ *satisfies* U-cap$_{p,\le}(d_{\mathrm{w}}(p))$ *for all* $p \in [1,\infty)$.

(iii) *The sequence* $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$ *satisfies* U-BCL$_p(d_{\mathrm{f}} - d_{\mathrm{w}}(p))$ *for all* $p \in (1,\infty)$.

Proof. (i) Since $d_{\mathrm{f}} < 2$ and $d_{\mathrm{w}}(p) \ge p$ (see [Shi24, Proposition 3.5] or [Kig20, Lemma 4.6.15]), we have $d_{\mathrm{f}} - d_{\mathrm{w}}(p) < 2 - p \le 1$ for all $p \ge 1$.

(ii) By virtue of a similar argument to the last part in Lemma 5.2, it is enough to estimate discrete p-capacities for large enough R, say $R \ge 2a_* + 1$. Let $n \in \mathbb{N}$, $x \in V_n$ and $R \in [2a_* + 1, \operatorname{diam}(\mathbb{G}_n))$. Let $n(R) \in \mathbb{Z}$ be the unique integer such that

$$2a_*^{n(R)} < R \le 2a_*^{n(R)+1}.$$

Then $1 \le n(R) < n$ since $R > 2a_*$ and $R \le 2a_*^n$.

For each $w \in V_{n(R)}$, let $\varphi_w \colon V_n \to [0,1]$ satisfy $\varphi_w|_{S^{n-n(R)}(w)} \equiv 1$, $\operatorname{supp}[\varphi_w] \subseteq \bigcup_{v\in V_{n(R)};d_{n(R)}(v,w)\le 1} S^{n-n(R)}(v)$ and $\mathcal{E}_p^{\mathbb{G}_n}(\varphi_w) = \mathrm{cap}_p^{\mathbb{G}_{n+m}}\left(S^{n-n(R)}(w), V_n \backslash S^{n-n(R)}(B_{d_{n(R)}}(w,2))\right)$. Let $\mathcal{N}(x,R) \coloneqq \left\{ w \in V_{n(R)} \;\middle|\; B_{d_n}(x,R) \cap S^{n-n(R)}(w) \neq \emptyset \right\}$. Since $\mathbb{G}_n$ is metric doubling and its doubling constant depends only on a, N_*, we easily see that $\#\mathcal{N}(x,R) \lesssim 1$, where the bound also depends only on a, N_*. Let $\varphi \coloneqq \sum_{w\in\mathcal{N}(x,R)} \varphi_w$. Then $\varphi|_{B_{d_n}(x,R)} \equiv 1$, $\operatorname{supp}[\varphi] \subseteq B_{d_n}(x,2R)$ and $\mathcal{E}_p^{\mathbb{G}_n}(\varphi) \le (\#\mathcal{N}(x,r))^{p-1}\mathcal{C}_p^{(n)} \lesssim \rho(p)^{-n}$. Since $\rho(p)^{-n} = a_*^{n(d_{\mathrm{f}} - d_{\mathrm{w}}(p))} \lesssim \#B_{d_n}(x,R)/R^{d_{\mathrm{w}}(p)}$, we obtain U-cap$_{p,\le}(d_{\mathrm{w}}(p))$.

(iii) Thanks to Theorem 8.5, we easily have $\mathrm{Mod}_p^{\mathbb{G}_n}\left(W_n[\ell_1], W_n[\ell_2]\right) \asymp \rho(p)^{-n}$, $n \in \mathbb{N}$. This estimate together with a similar argument as in the proof of [Kig23, Theorem 4.8] implies that the following is true: For any $p \ge 1$ and $L \ge 1$, there exists $c > 0$ (depending only on p, L, L_*) such that for any $k, n \in \mathbb{N}$ and $v, w \in V_n$ with $d_n(v,w) \le L$,

$$\mathrm{Mod}_p^{\mathbb{G}_{n+k}}\left(\{\theta \in \mathsf{Path}(S^k(v), S^k(w); \mathbb{G}_{k+n}) \mid \operatorname{diam}(\theta, d_{k+n}) \le 2La_*^k\}\right) \ge c\rho(p)^{-k}. \tag{8.8}$$

Since each cell $S^k(z)$ can be compared to a discrete ball by virtue of (6.3), we can easily obtain U-BCL$_p(d_{\mathrm{f}} - d_{\mathrm{w}}(p))$ from the uniform bound (8.8). □

Proof of Theorem 8.3(a) and (b). (a) is proved in Proposition 8.4. (b) follows from Propositions 8.8, 6.8 and 6.12. In particular, Assumption 6.15 holds. □

Since we have checked Assumption 6.15 for PSC, we can apply Theorem 6.22 and get a p-energy $\mathcal{E}_p^{\Gamma}$ on PSC. We next show the pre-self-similarity condition (Theorem 8.3(c)), which is important to improve $\mathcal{E}_p^{\Gamma}$, by using an *unfolding argument* inspired by [Hin13, Subsection 5.1]. This argument is long, so we divide it into several steps. First, we prove the following easy bound:

$$\rho(p)^n \sum_{w\in W_n} |f\circ F_w|_{\mathcal{F}_p}^p \lesssim |f|_{\mathcal{F}_p}^p \quad \text{for all } f \in L^p(K,m). \tag{8.9}$$

Here we regard $|\cdot|_{\mathcal{F}_p}$ as a $[0,\infty]$-valued functional defined on $L^p(K,m)$, which satisfies $|f|_{\mathcal{F}_p} < \infty$ if and only if $f \in \mathcal{F}_p$.

Proof of (8.9). Since m is the self-similar measure with the equal weights, we have $M_n(f\circ F_w)(v) = M_{n+m}f(wv)$ for $n,m \in \mathbb{N}$ and $w \in V_m$, $v \in V_n$. Therefore,

$$\rho(p)^n \sum_{w\in W_n} \widetilde{\mathcal{E}}_p^{(m)}(f\circ F_w) = \sum_{w\in W_n} \widetilde{\mathcal{E}}_{p,S^m(w)}^{(n+m)}(f) \le \widetilde{\mathcal{E}}_p^{(n+m)}(f),$$

which together with the weak monotonicity (Theorem 6.13) implies (8.9). □

The reverse inequality is much harder. By adapting the argument in [Hin13, §5.1], we will show

$$|f|_{\mathcal{F}_p}^p \lesssim \rho(p)^n \sum_{w\in W_n} |f\circ F_w|_{\mathcal{F}_p}^p \quad \text{for all } f \in \mathcal{F}_p^S, \tag{8.10}$$

where we set $\mathcal{F}_p^S \coloneqq \{ f \in \mathcal{C}(K) \mid f\circ F_i \in \mathcal{F}_p \cap \mathcal{C}(K) \text{ for all } i \in S\}$.

The following *folding maps* play a key role. For details, we refer to [BBKT, Section 2.2] or [Kig23, Section 4.3].

Definition 8.9 (Folding maps and unfolding operators)**.** (1) For $n \in \mathbb{N}$, let $\widehat{\varphi}_n \colon \mathbb{R} \to [0,\infty)$ be the periodic function with period $4a_*^{-n}$ such that

$$\widehat{\varphi}_n(t) = \begin{cases} t+1 & \text{for } t \in [-1,-1+2a_*^{-n}], \\ -t-1+4a_*^{-n} & \text{for } t \in [-1+2a_*^{-n}, -1+4a_*^{-n}]. \end{cases}$$

Define $\varphi^{(n)} \colon [-1,1]^2 \to [0,2a_*^{-n}]^2$ by

$$\varphi^{[n]}(x,y) \coloneqq \bigl(\widehat{\varphi}_n(x), \widehat{\varphi}_n(y)\bigr) \quad \text{for } (x,y) \in [-1,1]^2.$$

For $w \in V_n$, define $\varphi_w \colon K \to K_w$ by

$$\varphi_w(x) \coloneqq \left(\varphi^{[n]}\big|_{K_w}\right)^{-1} \left(\varphi^{[n]}(x)\right) \quad \text{for } x \in K.$$

(2) Let $E_n^{\#} := \bigl\{\{v,w\} \in E_n \bigm| \#(K_v \cap K_w) \geq 2\bigr\}$. For $\{v,w\} \in E_n^{\#}$, let $\ell_{v,w} := K_v \cap K_w$ and let $H_{v,w}$ be the line containing $\ell_{v,w}$. Then $H_{v,w}$ splits $\mathbb{R}^2$ into the two closed half spaces, which are denoted by $G_{v,w}$ and $G_{w,v}$ and satisfy $K_v \subseteq G_{v,w}$ and $K_w \subseteq G_{w,v}$. We remark that the order of v and w is important in the notations $G_{v,w}$, $G_{w,v}$.

(3) For $f \in L^p(K,m)$ and $w \in W_n$, define $\Xi_w(f) := f \circ \varphi_w$. The map Ξ_w is called an *unfolding operator*. For $\{v,w\} \in E_n^{\#}$, define $\Xi_{v,w}(f) := \Xi_v(f)\mathbb{1}_{G_{v,w}}$.

To provide a quantitative (localized) energy estimate for $\Xi_z(f)$ by following [Hin13], we make the help of Korevaar–Schoen type bounds given in Section 7. Recall that, by Theorem 7.1, there exists $C \geq 1$ such that, for all $f \in L^p(K,m)$,

$$C^{-1}|f|_{\mathcal{F}_p}^p \leq \varlimsup_{r\downarrow 0} r^{-d_{\mathrm{w}}(p)} J_{p,r}(f) \leq C|f|_{\mathcal{F}_p}^p. \tag{8.11}$$

Let us introduce some notations for simplicity. For $f \in L^p(K,m)$, $\delta > 0$, and $A_1, A_2 \in \mathcal{B}(K)$, define

$$E_{p,\delta}(f;A_1,A_2) := \delta^{-(d_{\mathrm{f}}+d_{\mathrm{w}}(p))} \iint_{\{(x,y)\in A_1\times A_2 \mid d(x,y)<\delta\}} |f(x)-f(y)|^p\, m(dx)m(dy).$$

We also write $E_{p,\delta}(f;A)$ and $E_{p,\delta}(f)$ for $E_{p,\delta}(f;A,A)$ and $E_{p,\delta}(f;K)$ respectively. Since m is the self-similar measure with the weight $(a_*^{-d_{\mathrm{f}}}, \dots, a_*^{-d_{\mathrm{f}}})$, we have

$$E_{p,\delta}(f;K_w) = \rho(p)^n E_{p,a_*^n\delta}(f \circ F_w) \quad \text{for any } w \in W_n.$$

Note that $\rho(p)^n a_*^{-n(d_{\mathrm{f}}+d_{\mathrm{w}}(p))} = a_*^{-2nd_{\mathrm{f}}}$.

The following estimate on localized energies of $\Xi_z(f)$ is a key ingredient, which corresponds to [Hin13, Corollary 5.4].

Lemma 8.10. *Let $n \in \mathbb{N}$, $z \in W_n$, $\delta > 0$ and $f \in L^p(K,m)$. Then, for any $\{v,w\} \in E_n$,*

$$E_{p,\delta}\bigl(\Xi_z(f);K_v,K_w\bigr) \leq E_{p,\delta}\bigl(\Xi_z(f);K_v\bigr) \leq \rho(p)^n E_{p,a_*^n\delta}(f \circ F_z).$$

In particular, there exists $C > 0$ such that

$$|\Xi_z(f)|_{\mathcal{F}_p}^p \leq C(\#W_n)\rho(p)^n|f \circ F_z|_{\mathcal{F}_p}^p \quad \textit{for all } f \in L^p(K,m),\ n \in \mathbb{N} \textit{ and } z \in W_n.$$

Proof. Using [BBKT, (2.22)] with $\nu = m|_{K_v}$ and following [Hin13, Lemma 5.3], we have that, for $v, z \in W_n$,

$$E_{p,\delta}\bigl(\Xi_z(f);K_v\bigr) = E_{p,\delta}(f;K_z) = \rho(p)^n E_{p,a_*^n\delta}(f \circ F_z),$$

and

$$E_{p,\delta}\bigl(\Xi_z(f);K_v,K_w\bigr) \leq E_{p,\delta}\bigl(\Xi_z(f);K_v\bigr).$$

Next we give an estimate for $|\Xi_z(f)|_{\mathcal{F}_p}$. Let $n \in \mathbb{N}$ and $z \in W_n$. For small enough $\delta > 0$,

$$E_{p,\delta}(\Xi_z(f)) = \sum_{v\in W_n} E_{p,\delta}(\Xi_z(f);K_v) + \sum_{\{v,w\}\in E_n} E_{p,\delta}(\Xi_z(f);K_v,K_w).$$

Therefore, we have $E_{p,\delta}(\Xi_z(f)) \leq (1+L_*)\rho(p)^n \sum_{v\in W_n} E_{p,a_*^n\delta}(f\circ F_z)$ (recall that $L_* = \sup_{n\in\mathbb{N}} \deg(\mathbb{G}_n) < \infty$ as defined in Definition 6.9). Hence

$$\overline{\lim_{\delta\downarrow 0}} E_{p,\delta}(\Xi_z(f)) \lesssim \rho(p)^n \overline{\lim_{\delta\downarrow 0}} E_{p,\delta}(f\circ F_z)(\#W_n),$$

which together with (8.11) yields the desired conclusion. □

We also need the following approximation.

Lemma 8.11. *Let F be a non-empty subset of K. Suppose that $f \in \mathcal{F}_p \cap \mathcal{C}(K)$ satisfies $f(x) = 0$ for all $x \in F$. Then there exist $f_n \in \mathcal{F}_p \cap \mathcal{C}(K)$ ($n \in \mathbb{N}$) such that $\operatorname{supp}[f_n] \subseteq K\setminus F$ for all $n \in \mathbb{N}$ and f_n converges in $\mathcal{F}_p$ to f as $n \to \infty$.*

Proof. We first consider the case that f is non-negative, i.e., let us suppose that $f \in \mathcal{F}_p \cap \mathcal{C}(K)$ satisfies $f\big|_F = 0$ and $f \geq 0$. Since f is uniformly continuous, for any $n \in \mathbb{N}$ there exists $r_n > 0$ such that $f(x) < n^{-1}$ for all $x \in F_n := \bigcup_{x\in F} B_d(x, r_n)$. Define $f_n \in \mathcal{F}_p \cap \mathcal{C}(K)$ by $f_n = \left(f - n^{-1}\right) \vee 0$. Then we immediately have $f_n(x) = 0$ for $x \in F_n$ and $\operatorname{supp}[f_n] \subseteq K \setminus F$. Furthermore, by Theorem 6.22(ii) (or (8.11)), we have $|f_n|_{\mathcal{F}_p} \leq C|f|_{\mathcal{F}_p}$ for any $n \in \mathbb{N}$, where C is independent of f and n. It is also clear that $\sup_{x\in K} |f(x) - f_n(x)| \to 0$ as $n \to \infty$ and hence $\|f - f_n\|_{L^p} \to 0$. Since $\{f_n\}_{n\geq 1}$ is a bounded sequence in $\mathcal{F}_p$, there exists a subsequence $\{f_{n_k}\}_{k\geq 1}$ such that f_{n_k} converges weakly in $\mathcal{F}_p$ to f as $k \to \infty$. Applying Mazur's lemma (Lemma A.1), there exists $g_n \in \mathcal{F}_p \cap \mathcal{C}(K)$ ($n \geq 1$) such that $\operatorname{supp}[g_n] \subseteq K \setminus F$ and $\|f - g_n\|_{\mathcal{F}_p} \to 0$, which proves our assertion.

For general $f \in \mathcal{F}_p \cap \mathcal{C}(K)$ satisfying $f\big|_F = 0$, we obtain the assertion by applying the above result for $f^{\pm}$. □

Next we prove a Fatou type lemma for localized Korevaar–Schoen energies.

Lemma 8.12. *Let $f, f_k \in \mathcal{F}_p$, $k \in \mathbb{N}$, such that f_k converges in $L^p(K,m)$ to f as $k \to \infty$. Suppose $\sup_{k\in\mathbb{N}} |f_k|_{\mathcal{F}_p} < \infty$. Then, for any $n \in \mathbb{N}$ and $\{v, w\} \in E_n$,*

$$\limsup_{\delta\downarrow 0} E_{p,\delta}(f; K_v, K_w) \leq \liminf_{n\to\infty} \limsup_{\delta\downarrow 0} E_{p,\delta}(f_n; K_v, K_w).$$

Proof. First, we prove the following claim: for any $g, g_k \in \mathcal{F}_p$, $k \in \mathbb{N}$, such that $\lim_{k\to\infty} |g - g_k|_{\mathcal{F}_p} = 0$, we have

$$\lim_{k\to\infty} \overline{\lim_{\delta\downarrow 0}} E_{p,\delta}(g_k; K_v, K_w) = \overline{\lim_{\delta\downarrow 0}} E_{p,\delta}(g; K_v, K_w). \tag{8.12}$$

This is immediate since

$$\begin{aligned} \left| \overline{\lim_{\delta\downarrow 0}} E_{p,\delta}(g; K_v, K_w)^{1/p} - \overline{\lim_{\delta\downarrow 0}} E_{p,\delta}(g_k; K_v, K_w)^{1/p} \right| &\leq \overline{\lim_{\delta\downarrow 0}} E_{p,\delta}(g - g_n; K_v, K_w)^{1/p} \\ &\leq \overline{\lim_{\delta\downarrow 0}} E_{p,\delta}(g - g_k)^{1/p} \lesssim |g - g_k|_{\mathcal{F}_p}. \end{aligned}$$

The rest of the proof is a standard argument using Mazur's lemma (Lemma A.1). Let $f_k \in \mathcal{F}_p$, $k \in \mathbb{N}$ be a sequence converging in L^p to some $f \in \mathcal{F}_p$. By extracting a subsequence $\{f_{k'}\}_{k'}$ if necessary, we can assume that

$$\lim_{k'\to\infty} \varlimsup_{\delta\downarrow 0} E_{p,\delta}(f_{k'}; K_v, K_w) = \varliminf_{k\to\infty} \varlimsup_{\delta\downarrow 0} E_{p,\delta}(f_k; K_v, K_w).$$

Since $\mathcal{F}_p$ is reflexive, there exists a subsequence, which is also denoted by $\{f_{k'}\}_{k'}$, such that $f_{k'}$ converges weakly in $\mathcal{F}_p$ to f. By Mazur's lemma, there exist finite subset $I_j \subseteq [j,\infty) \cap \mathbb{N}$, $j \in \mathbb{N}$, and

$$\left\{ \lambda_{k'}^{(j)} \;\middle|\; \lambda_{k'}^{(j)} \geq 0 \text{ for } k' \in I_j \text{ and } \sum_{k' \in I_j} \lambda_{k'}^{(j)} = 1 \right\}_{j \in \mathbb{N}}$$

such that $g_j := \sum_{k' \in I_j} \lambda_{k'}^{(j)} f_{k'} \in \mathcal{F}_p$ $(j \in \mathbb{N})$ satisfies $\|f - g_j\|_{\mathcal{F}_p} \to 0$ as $j \to \infty$. By the triangle inequality of L^p-norm, we see that

$$\varlimsup_{\delta\downarrow 0} E_{p,\delta}(g_j; K_v, K_w)^{1/p} \leq \sum_{k' \in I_j} \lambda_{k'}^{(j)} \varlimsup_{\delta\downarrow 0} E_{p,\delta}(f_{k'}; K_v, K_w)^{1/p}.$$

Letting $j \to \infty$ and using (8.12), we obtain

$$\varlimsup_{\delta\downarrow 0} E_{p,\delta}(f; K_v, K_w)^{1/p} \leq \lim_{k'\to\infty} \varlimsup_{\delta\downarrow 0} E_{p,\delta}(f_{k'}; K_v, K_w)^{1/p},$$

proving our assertion. □

Now we can estimate the unfolding map $\Xi_{v,w}(f)$ for $\{v, w\} \in E_n^{\#}$.

Lemma 8.13. *Let $n \in \mathbb{N}$, $\{v, w\} \in E_n^{\#}$ and $f \in \mathcal{F}_p^S$. If $f\big|_{\ell_{v,w}} = 0$ and $z \in W_n$ satisfies $K_z \subseteq G_{w,v}$, then $\lim_{\delta\downarrow 0} E_{p,\delta}\bigl(\Xi_{v,w}(f); K_v, K_z\bigr) = 0$.*

Proof. This lemma corresponds to a weaker version of [Hin13, Lemma 5.6]. Let $n \in \mathbb{N}$ and $\{v, w\} \in E_n^{\#}$. Let $f \in \mathcal{F}_p^S$ satisfy $f\big|_{\ell_{v,w}} = 0$. Note that $\Xi_v(f) \in \mathcal{F}_p \cap \mathcal{C}(K)$ by Lemma 8.10. Applying Lemma 8.11 for $\Xi_v(f)$, we obtain a sequence $f_k \in \mathcal{F}_p \cap \mathcal{C}(K)$, $k \in \mathbb{N}$ such that $\mathrm{supp}[f_k] \subseteq K \setminus \ell_{v,w}$ and f_k converges in $\mathcal{F}_p$ to $\Xi_v(f)$. Set $g_k := \Xi_v(f_k)$ and $h_k := \Xi_{v,w}(f_k)$ for $k \geq 1$. For $\delta < \mathrm{dist}_d(H_{v,w}, \mathrm{supp}[g_k])$, we see that

$$E_{p,\delta}(h_k) = E_{p,\delta}(g_k; K \cap G_{v,w}) \leq E_{p,\delta}(g_k).$$

Combining with Lemma 8.10 and (8.11), we obtain

$$|h_k|_{\mathcal{F}_p}^p \lesssim \varlimsup_{\delta\downarrow 0} E_{p,\delta}(h_k) \leq \varlimsup_{\delta\downarrow 0} E_{p,\delta}(g_k) \leq C(\#W_n)\rho(p)^n |f_k \circ F_v|_{\mathcal{F}_p}^p,$$

which together with (8.9) implies that $\{h_k\}_{k\geq 1}$ is bounded in $\mathcal{F}_p$. Note that h_k converges in $L^p(K, m)$ to $\Xi_{v,w}(f)$ as $k \to \infty$. Hence, by Lemma 8.12, for all $z \in V_n$ such that $K_z \subseteq G_{w,v}$,

$$\varlimsup_{\delta\downarrow 0} E_{p,\delta}(\Xi_{v,w}(f); K_v, K_z) \leq \varliminf_{k\to\infty} \varlimsup_{\delta\downarrow 0} E_{p,\delta}(h_k; K_v, K_z).$$

If $\delta < \mathrm{dist}_d(H_{v,w}, \mathrm{supp}[g_k])$, then we have $E_{p,\delta}(h_k; K_v, K_z) = 0$. Therefore, we obtain $\overline{\lim}_{\delta\downarrow 0} E_{p,\delta}(\Xi_{v,w}(f); K_v, K_z) = 0$. This completes the proof. $\square$

Finally, we can prove the bound (8.10) and complete the proof of Theorem 8.3(c).

Proof of Theorem 8.3(c). The estimate (8.9) is already proved. In particular, $\mathcal{F}_p = \{f \in \mathcal{F}_p \mid f \circ F_i \in \mathcal{F}_p \text{ for all } i \in S\}$. To prove (8.10), let $f \in \mathcal{F}_p^S$. Let us fix $n \in \mathbb{N}$. Then, for small enough $\delta > 0$, we observe that

$$E_{p,\delta}(f) = \sum_{w \in W_n} E_{p,\delta}(f; K_w) + \sum_{\{v,w\} \in E_n} E_{p,\delta}(f; K_v, K_w). \tag{8.13}$$

We obtain upper bounds for $E_{p,\delta}(f; K_v, K_w)$ by dividing into the following two cases.
Case 1: $\{v,w\} \in E_n^{\#}$; Define $h_i \in \mathcal{C}(K)$, $i \in \{0,1\}$, by

$$h_0 \coloneqq \Xi_v(f) \quad \text{and} \quad h_1 \coloneqq \Xi_{w,v}(f - h_0).$$

It is easy to see that $f\big|_{K_v \cup K_w} = (h_0 + h_1)\big|_{K_v \cup K_w}$ and that $(f - h_0)\big|_{\ell_{v,w}} = 0$. Since $h_0 \in \mathcal{F}_p \cap \mathcal{C}(K)$ by Lemma 8.10 and $f \in \mathcal{F}_p^S$, it is also immediate that $f - h_0 \in \mathcal{F}_p^S$. Hence, by Lemmas 8.10 and 8.13,

$$\begin{aligned}
\overline{\lim_{\delta\downarrow 0}} E_{p,\delta}(f; K_v, K_w) &\le 2^{p-1} \overline{\lim_{\delta\downarrow 0}} \bigl(E_{p,\delta}(h_0; K_v, K_w) + E_{p,\delta}(h_1; K_v, K_w)\bigr) \\
&\le 2^{p-1} \rho(p)^n \overline{\lim_{\delta\downarrow 0}} E_{p,\delta}(f \circ F_v).
\end{aligned}$$

Case 2: $\{v,w\} \in E_n \setminus E_n^{\#}$; Clearly, there exists $z(i) \in W_n$, $i \in \{1,2,3\}$, such that $\{z(1), z(3)\} = \{v,w\}$, $\{z(i), z(i+1)\} \in E_n^{\#}$ for $i \in \{1,2\}$ and $K_{z(i)} \not\subseteq G_{z(j),z(2)}$ for $\{i,j\} = \{1,3\}$. Now we define $h_i \in \mathcal{C}(K)$, $i \in \{0,1,2\}$, by

$$h_0 \coloneqq \Xi_{z(2)}(f), \quad h_1 \coloneqq \Xi_{z(1),z(2)}(f - h_0) \quad \text{and} \quad h_2 \coloneqq \Xi_{z(3),z(2)}(f - h_0).$$

Then we have $f\big|_{\cup_{i=1}^3 K_{z(i)}} = (h_0 + h_1 + h_2)\big|_{\cup_{i=1}^3 K_{z(i)}}$ and $(f - h_0)\big|_{\ell_{z(1),z(2)} \cup \ell_{z(2),z(3)}} = 0$. Hence, by Lemmas 8.10 and 8.13,

$$\overline{\lim_{\delta\downarrow 0}} E_{p,\delta}(f; K_v, K_w) \le 3^{p-1} \overline{\lim_{\delta\downarrow 0}} \sum_{j=0}^{2} E_{p,\delta}(h_j; K_v, K_w) \le 3^{p-1} \rho(p)^n \overline{\lim_{\delta\downarrow 0}} E_{p,\delta}\bigl(f \circ F_{z(2)}\bigr).$$

From (8.13) and above observations, we obtain

$$\overline{\lim_{\delta\downarrow 0}} E_{p,\delta}(f) \le (1 + L_*^2) \rho(p)^n \sum_{v \in W_n} \overline{\lim_{\delta\downarrow 0}} E_{p,\delta}(f \circ F_v),$$

which together with (8.11) proves (8.10) (recall that $L_* = \sup_{n \in \mathbb{N}} \deg(\mathbb{G}_n) < \infty$ as defined in Definition 6.9). Note that (8.10) implies $\mathcal{F}_p^S = \mathcal{F}_p \cap \mathcal{C}(K)$. We finish the proof of Theorem 8.3. $\square$

We are now ready to prove the first four main results stated in the introduction.

Proof of Theorem 1.1. Theorem 8.3 implies Assumptions 6.15. Therefore by Theorem 6.17 we obtain the conclusions (i) and (ii).

The existence of a self-similar energy with the desired properties follows from a standard argument; see [Kig00, Theorem 1.5] or [KS24+b, Theorem 5.22] for example. We shall briefly explain how to obtain the desired p-energy functional. We set

$$\mathcal{U}_p := \left\{ \mathcal{E}\colon \mathcal{F}_p \to [0,\infty) \;\middle|\; \begin{array}{c} \mathcal{E}^{1/p} \text{ is a seminorm on } \mathcal{F}_p, \text{ there exist } c_1, c_2 > 0 \\ \text{such that } c_1 |f|_{\mathcal{F}_p} \le \mathcal{E}(f)^{1/p} \le c_2 |f|_{\mathcal{F}_p} \text{ for any } f \in \mathcal{F}_p \end{array} \right\},$$

and define $\mathcal{S}\colon \mathcal{U}_p \to \mathcal{U}_p$ by

$$\mathcal{S}E(f) := \rho(p) \sum_{i \in S} E(f \circ F_i), \quad E \in \mathcal{U}_p, f \in \mathcal{F}_p.$$

By (8.2), there exists $C \ge 1$ such that

$$C^{-1} |f|_{\mathcal{F}_p}^p \le \mathcal{S}^n \mathcal{E}_p^\Gamma(f) = \rho(p)^n \sum_{w \in W_n} \mathcal{E}_p^\Gamma(f \circ F_w) \le C |f|_{\mathcal{F}_p}^p \quad \text{for any } f \in \mathcal{F}_p \text{ and } n \in \mathbb{N},$$

where $\mathcal{E}_p^\Gamma \in \mathcal{U}_p$ is given by Theorem 6.22. Since $\mathcal{F}_p$ is reflexive and separable by Theorem 6.17, there exists $\{n_k\}_{k \in \mathbb{N}} \subseteq \mathbb{N}$ with $n_k < n_{k+1}$ for any $k \in \mathbb{N}$ such that the following limit exists in $[0,\infty)$ for any $f \in \mathcal{F}_p$:

$$\mathcal{E}_p(f) := \lim_{k \to \infty} \frac{1}{n_k} \sum_{j=0}^{n_k - 1} \mathcal{S}^j \mathcal{E}_p^\Gamma(f). \tag{8.14}$$

Then it is easy to see that $\mathcal{E}_p$ has the desired self-similarity (1.2), which together with (8.1) in Theorem 8.3 shows (viii). The other properties (iii), (iv), (v) and (ix) are easily implied by Theorem 6.22 and the explicit expression (8.14) (note that $\widetilde{\mathcal{E}}_p^{(n)}(f \circ T) = \widetilde{\mathcal{E}}_p^{(n)}(f)$ for any $n \in \mathbb{N}$, $f \in L^p(K, m)$ and $T \in D_4$ by Proposition 8.4(vi) and the self-similarity of m).

The strong locality (vii) is an easy consequence of the self-similarity (viii) (see [Shi24, Subsection 6.2]). The estimate in (vi) follows from applying Lemma 6.24 with $r = 2\operatorname{diam}(K, d)$. The proof is completed. □

Proof of Theorem 1.4. As mentioned earlier, Assumption 6.15 follows from Theorem 8.3. The desired conclusion then follows from any application of Theorem 7.1. □

8.2 Associated energy measures: Theorem 1.2 and 1.5

We next focus on the object called *energy measure* associated with our self-similar p-energy. Hereafter, we let $\mathcal{E}_p$ be a p-energy as given in Theorem 1.1. Then we can introduce measures by using the self-similarity of $\mathcal{E}_p$ as done in [Shi24, Section 7] (see [Hin05, Lemma 4.1] for the case $p = 2$). Let $f \in \mathcal{F}_p$ and $n \in \mathbb{Z}_{\geq 0}$. Define a finite measure $\mathfrak{m}_p^{(n)}\langle f \rangle$ on W_n by setting $\mathfrak{m}_p^{(n)}\langle f \rangle(\{w\}) := \rho(p)^n \mathcal{E}_p(f \circ F_w)$ for each $w \in W_n$. Due to the following equalities:

$$\sum_{v \in S(w)} \mathfrak{m}_p^{(n+1)}\langle f \rangle(\{v\}) = \rho(p)^n \sum_{i \in S} \rho(p) \mathcal{E}_p\bigl((f \circ F_w) \circ F_i\bigr) = \mathfrak{m}_p^{(n)}\langle f \rangle(\{w\}),$$

we can use Kolmogorov's extension theorem (see [Dud, Theorem 12.1.2] for example) to get a finite Borel measure $\mathfrak{m}_p\langle f \rangle$ on $\Sigma = S^{\mathbb{N}}$ such that

$$\mathfrak{m}_p\langle f \rangle(\Sigma_w) = \rho(p)^n \mathcal{E}_p(f \circ F_w) \quad \text{for any } n \in \mathbb{Z}_{\geq 0} \text{ and } w \in W_n.$$

Clearly, $\mathfrak{m}_p\langle f \rangle(\Sigma) = \mathcal{E}_p(f)$.

Now we define a measure $\Gamma_p\langle f \rangle$ on K as $\Gamma_p\langle f \rangle := \chi_*(\mathfrak{m}_p\langle f \rangle)$, where χ is the map in Definition 8.2. Note that $\Gamma_p\langle f \rangle$ is a finite Borel-regular measure on K (see [Dud, Theorem 7.1.3] for example). We shall say that $\Gamma_p\langle f \rangle$ is the *$\mathcal{E}_p$-energy measure of f*.

The following lemma gives behaviors of 'p-energy on each cells'.

Lemma 8.14. *For any $f \in \mathcal{F}_p$, $w \in W_*$ and $n \in \mathbb{Z}_{\geq 0}$,*

$$\rho(p)^n \mathcal{E}_p(f \circ F_w) \leq \Gamma_p\langle f \rangle(K_w) \leq \rho(p)^n \sum_{v \in W_n[K_w]} \mathcal{E}_p(f \circ F_v).$$

Proof. The lower bound is immediate from $\Sigma_w \subseteq \chi^{-1}(K_w)$. The upper bound follows from $\chi^{-1}(K_w) \subseteq \bigcup_{v \in W_n[K_w]} \Sigma_v$. □

The following proposition is a collection of basic properties of energy measures.

Proposition 8.15. (a) *Let $f \in \mathcal{F}_p$. Then $\Gamma_p\langle f \rangle \equiv 0$ if and only if f is constant.*

(b) *For any $f, g \in \mathcal{F}_p$ and $A \in \mathcal{B}(K)$,*

$$\Gamma_p\langle f + g \rangle(A)^{1/p} \leq \Gamma_p\langle f \rangle(A)^{1/p} + \Gamma_p\langle g \rangle(A)^{1/p}. \tag{8.15}$$

(c) *If $f \in \mathcal{F}_p$ and $\varphi \in \mathcal{C}(\mathbb{R})$ is 1-Lipschitz, then $\Gamma_p\langle \varphi \circ f \rangle(A) \leq \Gamma_p\langle f \rangle(A)$ for any $A \in \mathcal{B}(K)$.*

(d) *For any $n \in \mathbb{N}$ and $f \in \mathcal{F}_p$,*

$$\Gamma_p\langle f \rangle = \rho(p)^n \sum_{w \in W_n} (F_w)_* \Gamma_p\langle f \circ F_w \rangle, \tag{8.16}$$

that is, $\Gamma_p\langle f \rangle(A) = \rho(p)^n \sum_{w \in W_n} \Gamma_p\langle f \circ F_w \rangle\bigl(F_w^{-1}(A)\bigr)$ for all $A \in \mathcal{B}(K)$.

(e) *For any* $\Phi \in D_4$ *and* $f \in \mathcal{F}_p$, *we have* $\Phi_*\bigl(\Gamma_p\langle f\rangle\bigr) = \Gamma_p\langle f \circ \Phi\rangle$.

Proof. (a) It is clear from $\Gamma_p\langle f\rangle(K) = \mathcal{E}_p(f)$ and (1.1).

(b) It suffices to show (8.15) when A is a closed set of K since $\Gamma_p\langle f+g\rangle$ is Borel regular. For $n \in \mathbb{N}$, define $C_n \coloneqq \{w \in W_n \mid \Sigma_w \cap \chi^{-1}(A) \neq \emptyset\}$ and $\Sigma_{C_n} \coloneqq \{\omega \in \Sigma(S) \mid [\omega]_n \in C_n\}$. Then $\bigl\{\Sigma_{C_n}\bigr\}_{n\geq 1}$ is a decreasing sequence satisfying $\bigcap_{n\in\mathbb{N}} \Sigma_{C_n} = \chi^{-1}(A)$ (see the proof of [Hin05, Lemma 4.1]). By the triangle inequality,

$$\begin{aligned}\left(\sum_{w\in C_n} \mathcal{E}_p\bigl((f+g)\circ F_w\bigr)\right)^{1/p} &\leq \left(\sum_{w\in C_n} \Bigl(\mathcal{E}_p(f\circ F_w)^{1/p} + \mathcal{E}_p(g\circ F_w)^{1/p}\Bigr)^p\right)^{1/p} \\ &\leq \left(\sum_{w\in C_n} \mathcal{E}_p(f\circ F_w)\right)^{1/p} + \left(\sum_{w\in C_n} \mathcal{E}_p(g\circ F_w)\right)^{1/p},\end{aligned}$$

which proves $\mathfrak{m}_p\langle f+g\rangle\bigl(\Sigma_{C_n}\bigr)^{1/p} \leq \mathfrak{m}_p\langle f\rangle\bigl(\Sigma_{C_n}\bigr)^{1/p} + \mathfrak{m}_p\langle g\rangle\bigl(\Sigma_{C_n}\bigr)^{1/p}$. Letting $n \to \infty$ yields (8.15) for any closed set A.

(c) By Theorem 1.1(v), $\rho(p)^n \sum_{w\in A} \mathcal{E}_p\bigl((\varphi\circ f)\circ F_w\bigr) \leq \rho(p)^n \sum_{w\in A} \mathcal{E}_p\bigl(f\circ F_w\bigr)$ for any $n \in \mathbb{N}$ and $A \subseteq W_n$. A similar approximation argument in (b) proves the assertion.

(d) The proof is exactly the same as in [Shi24, Theorem 7.5].

(e) By the Borel regularity, it suffices to show the case that A is a closed set of K. Let $f \in \mathcal{F}_p$ and $\Phi \in D_4$. For each $n \in \mathbb{N}$, define $C_n \coloneqq \bigl\{w \in W_n \bigm| \Sigma_w \cap \chi^{-1}(A) \neq \emptyset\bigr\}$, $C_{n,\Phi} \coloneqq \bigl\{w \in W_n \bigm| \Sigma_w \cap \chi^{-1}(\Phi(A)) \neq \emptyset\bigr\}$, $\Sigma_{C_n} \coloneqq \bigl\{\omega \in \Sigma \bigm| [\omega]_n \in C_n\bigr\}$ and $\Sigma_{C_{n,\Phi}} \coloneqq \bigl\{\omega \in \Sigma \bigm| [\omega]_n \in C_{n,\Phi}\bigr\}$. Then $\tau_\Phi|_{C_n}$ gives a bijection between C_n and $C_{n,\Phi}$. Hence

$$\begin{aligned}\mathfrak{m}_p\langle f\circ\Phi\rangle\bigl(\Sigma_{C_n}\bigr) &= \rho(p)^n \sum_{w\in C_n} \mathcal{E}_p\bigl(f\circ(\Phi\circ F_w)\bigr) = \rho(p)^n \sum_{w\in C_n} \mathcal{E}_p\bigl((f\circ F_{\tau_\Phi(w)})\circ U_{\Phi,w}\bigr) \\ &= \rho(p)^n \sum_{w\in C_n} \mathcal{E}_p\bigl(f\circ F_{\tau_\Phi(w)}\bigr) = \rho(p)^n \sum_{v\in C_{n,\Phi}} \mathcal{E}_p(f\circ F_v) = \mathfrak{m}_p\langle f\rangle\bigl(\Sigma_{C_{n,\Phi}}\bigr).\end{aligned}$$

Letting $n \to \infty$, we obtain $\Gamma_p\langle f\circ\Phi\rangle(A) = \Phi_*\bigl(\Gamma_p\langle f\rangle\bigr)(A)$ since $\bigcap_{n\in\mathbb{N}} \Sigma_{C_n} = \chi^{-1}(A)$ and $\bigcap_{n\in\mathbb{N}} \Sigma_{C_{n,\Phi}} = \chi^{-1}(\Phi^{-1}(A))$. Hence we obtain $\Phi_*\bigl(\Gamma_p\langle f\rangle\bigr)(A) = \Gamma_p\langle f\circ\Phi\rangle(A)$ for any closed set A of K. □

Let us move to the *chain rule* of energy measures. The following 'weak locality' of energy measures corresponds to the condition (H5) in [BV05], which is a consequence of the self-similarity of energies.

Lemma 8.16. *Let* U *be an open subset of* K. *If* $f, g \in \mathcal{F}_p$ *satisfy* $f = g$ m*-a.e. on* U, *then* $\Gamma_p\langle f\rangle(U) = \Gamma_p\langle g\rangle(U)$.

Proof. By the inner regularity of $\Gamma_p\langle f\rangle$ and $\Gamma_p\langle g\rangle$, it suffices to show $\Gamma_p\langle f\rangle(A) = \Gamma_p\langle g\rangle(A)$ for any closed subset A of U. Pick $\delta \in (0, \mathrm{dist}_d(A, K\setminus U))$ and $N \in \mathbb{N}$ so that $a_*^{-n} < \delta$

for any $n \geq N$. For $n \in \mathbb{N}$, define $C_n := \{w \in V_n \mid \Sigma_w \cap \chi^{-1}(A) \neq \emptyset\}$. Since $f \circ F_w = g \circ F_w$ (m-a.e. on K) for any $w \in C_n$ with $n \geq N$, we have

$$\mathfrak{m}_p\langle f\rangle(\Sigma_{C_n}) = \rho(p)^n \sum_{w \in C_n} \mathcal{E}_p(f \circ F_w) = \rho(p)^n \sum_{w \in C_n} \mathcal{E}_p(g \circ F_w) = \mathfrak{m}_p\langle g\rangle(\Sigma_{C_n}).$$

Letting $n \to \infty$ proves $\Gamma_p\langle f\rangle(A) = \Gamma_p\langle g\rangle(A)$, which completes the proof. □

Now we show the chain rule for functions in $\mathcal{F}_p \cap \mathcal{C}(K)$.

Theorem 8.17 (Chain rule)**.** *For any $\Psi \in C^1(\mathbb{R})$ and $f \in \mathcal{F}_p \cap \mathcal{C}(K)$,*

$$\Gamma_p\langle \Psi \circ f\rangle(dx) = |\Psi'(f(x))|^p \Gamma_p\langle f\rangle(dx), \tag{8.17}$$

that is, $\Gamma_p\langle \Psi \circ f\rangle(A) = \int_A |\Psi'(f(x))|^p\, \Gamma_p\langle f\rangle(dx)$ for any $A \in \mathcal{B}(K)$.

Proof. The idea is very similar to [BV05, Proposition 4.1]. We present a complete proof because the framework of [BV05] is slightly different from our setting. Let $f \in \mathcal{F}_p \cap \mathcal{C}(K)$, $\Psi \in C^1(\mathbb{R})$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$|\Psi'(f(x)) - \Psi'(f(y))| < \varepsilon \quad \text{for any } x, y \in K \text{ with } d(x, y) < \delta.$$

Let $\{x_j\}_{j \in J}$ be a family such that $x_j \in K$ ($j \in J$), $\#J < \infty$ and $K = \bigcup_{j \in J} B_d(x_j, \delta)$. For $j \in J$, we define $\Psi_j \colon \mathbb{R} \to \mathbb{R}$ by

$$\Psi_j(t) = \frac{\Psi(f(x_j))}{|\Psi'(f(x_j))| + \varepsilon} + \int_{f(x_j)}^{t} \left[\left(\frac{\Psi'(s)}{|\Psi'(f(x_j))| + \varepsilon} \wedge 1\right) \vee (-1)\right] ds.$$

Then, it is clear than $\Psi_j \in C^1(\mathbb{R})$ and $\left|\Psi_j'(t)\right| \leq 1$ for all $t \in \mathbb{R}$. We note that if $s \in \mathbb{R}$ satisfies $|\Psi'(s) - \Psi'(f(x_j))| \leq \varepsilon$, then

$$\left(\frac{\Psi'(s)}{|\Psi'(f(x_j))| + \varepsilon} \wedge 1\right) \vee (-1) = \frac{\Psi'(s)}{|\Psi'(f(x_j))| + \varepsilon}.$$

In particular,

$$\Psi_j(f(x)) = \frac{\Psi(f(x))}{|\Psi'(f(x_j))| + \varepsilon} \quad \text{and} \quad \Psi_j'(f(x)) = \frac{\Psi'(f(x))}{|\Psi'(f(x_j))| + \varepsilon} \quad \text{for any } x \in B_d(x_j, \delta).$$

Set $a_j = |\Psi'(f(x_j))| + \varepsilon$ for simplicity. By Lemma 8.16, Proposition 8.15(d) and the outer regularity of energy measures, for any $E \in \mathcal{B}(K)$ with $E \subseteq B_d(x_j, \delta)$, we see that

$$\Gamma_p\langle \Psi \circ f\rangle(E) = \Gamma_p\big\langle a_j(\Psi_j \circ f)\big\rangle(E) = a_j^p \Gamma_p\langle \Psi_j \circ f\rangle(E) \leq \big(|\Psi'(f(x_j))| + \varepsilon\big)^p \Gamma_p\langle f\rangle(E).$$

Therefore, for $E \in \mathcal{B}(K)$ with $E \subseteq B_d(x_j, \delta)$,

$$\begin{aligned}
\Gamma_p\langle \Psi \circ f\rangle(E) &\leq \int_E |\Psi'(f(x))|^p\, \Gamma_p\langle f\rangle(dx) + \int_E \Big[\big(|\Psi'(f(x_j))| + \varepsilon\big)^p - |\Psi'(f(x))|^p\Big]\, \Gamma_p\langle f\rangle(dx) \\
&\leq \int_E |\Psi'(f(x))|^p\, \Gamma_p\langle f\rangle(dx) + \int_E \left|\int_{|\Psi'(f(x))|}^{|\Psi'(f(x_j))| + \varepsilon} p s^{p-1}\, ds\right| \Gamma_p\langle f\rangle(dx) \\
&\leq \int_E |\Psi'(f(x))|^p\, \Gamma_p\langle f\rangle(dx) + \varepsilon \cdot C_{p,\Psi,f} \Gamma_p\langle f\rangle(E),
\end{aligned} \tag{8.18}$$

where $C_{p,\Psi,f}$ is a constant depending only on p and $\sup_{t \in f(K)} |\Psi'(t)|$.

Now let $A \in \mathcal{B}(K)$ and let $J = \{1, \dots, N\}$. We inductively define A_j by $A_1 := A \cap B_d(x_1, \delta)$ and $A_{j+1} := \bigl(A \cap B_d(x_{j+1}, \delta)\bigr) \setminus A_j$ so that $A = \bigsqcup_{j=1}^{N} A_j$. By summing (8.18) with $E = A_j$ over j and letting $\varepsilon \downarrow 0$, we obtain

$$\Gamma_p\langle \Psi \circ f\rangle(A) \le \int_A |\Psi'(f(x))|^p \, \Gamma_p\langle f\rangle(dx) \quad \text{for any } A \in \mathcal{B}(K). \tag{8.19}$$

Next, we prove the converse inequality of (8.19). For $n \in \mathbb{N}$, we define a closed set F_n of K by $F_n := \bigl\{x \in K \bigm| |\Psi'(f(x))| \ge n^{-1}\bigr\}$. Note that $\bigcup_{n \ge 1} F_n = \{\Psi' \circ f \neq 0\}$. For each $n \in \mathbb{N}$ there exists $\delta_n > 0$ such that

$$|\Psi'(f(x)) - \Psi'(f(y))| < \frac{1}{2n} \quad \text{for any } x, y \in K \text{ with } d(x, y) < \delta_n.$$

Pick $l_n \in \mathbb{N}$ so that $\max_{w \in W_{l_n}} \operatorname{diam}(K_w, d) < \delta_n$. Let

$$F_n^+ := \bigl\{x \in K \bigm| \Psi'(f(x)) \ge n^{-1}\bigr\} = (\Psi' \circ f)^{-1}\Bigl(\bigl[n^{-1}, \infty\bigr)\Bigr),$$

$$F_n^- := \bigl\{x \in K \bigm| \Psi'(f(x)) \le -n^{-1}\bigr\} = (\Psi' \circ f)^{-1}\Bigl(\bigl(-\infty, -n^{-1}\bigr]\Bigr),$$

and $W_{l_n}[F_n^{\pm}] := \{w \in W_{l_n} \mid K_w \cap F_n^{\pm} \neq \emptyset\}$. Then, we easily see that

$$F_n = F_n^+ \sqcup F_n^- \subseteq \left(\bigcup_{w \in W_{l_n}[F_n^+]} K_w\right) \cup \left(\bigcup_{w \in W_{l_n}[F_n^-]} K_w\right),$$

and $\Psi'(f(y)) \ge (2n)^{-1}$ (resp. $\Psi'(f(y)) \le -(2n)^{-1}$) for any $y \in \bigcup_{w \in W_{l_n}[F_n^+]} K_w$ (resp. $y \in \bigcup_{w \in W_{l_n}[F_n^+]} K_w$). Since $f(K_w)$ is a connected subset of $\mathbb{R}$ and both functions f and $\Psi' \circ f$ are uniformly continuous on K, we can pick $\delta_n' > 0$ and a collection of open intervals $\{I_w\}_{w \in W_{l_n}[F_n^{\pm}]}$ so that

$$f\bigl((K_w)_{\delta_n'}\bigr) \subseteq I_w \text{ and } \inf_{t \in I_w} |\Psi'(t)| > 0 \text{ for any } w \in W_{l_n}[F_n^+] \sqcup W_{l_n}[F_n^-].$$

Since $\Psi \in C^1(\mathbb{R})$, Ψ' is strictly increasing or strictly decreasing on each I_w. Applying the inverse function theorem (e.g. [Jost, Theorem 2.7]), we get the inverse functions $\Upsilon_w \colon \Psi(I_w) \to \mathbb{R}$ of Ψ. For any $w \in W_{l_n}[F_n^+] \sqcup W_{l_n}[F_n^-]$ and any $E \in \mathcal{B}(K)$ with $E \subseteq K_w$, by Lemma 8.16 and the inequality (8.19) as measures,

$$\begin{aligned}
\int_E |\Psi'(f(x))|^p \, \Gamma_p\langle f\rangle(dx) &= \int_E |\Psi'(f(x))|^p \, \Gamma_p\langle \Upsilon_w \circ \Psi \circ f\rangle(dx) \\
&\le \int_E |\Upsilon_w'(\Psi(f(x)))|^p |\Psi'(f(x))|^p \, \Gamma_p\langle \Psi \circ f\rangle(dx) \\
&= \int_E d\Gamma_p\langle \Psi \circ f\rangle = \Gamma_p\langle \Psi \circ f\rangle(E).
\end{aligned}$$

A similar covering argument as in the previous paragraph yields, for any $A \in \mathcal{B}(K)$,

$$\int_{A\cap F_n} |\Psi'(f(x))|^p \,\Gamma_p\langle f\rangle(dx) \le \Gamma_p\langle \Psi\circ f\rangle(A\cap F_n).$$

By letting $n\to\infty$, we get

$$\begin{aligned}\int_A |\Psi'(f(x))|^p \,\Gamma_p\langle f\rangle(dx) &= \int_{A\cap\{\Psi'\circ f\neq 0\}} |\Psi'(f(x))|^p \,\Gamma_p\langle f\rangle(dx)\\ &\le \Gamma_p\langle \Psi\circ f\rangle(A\cap\{\Psi'\circ f\neq 0\}) \le \Gamma_p\langle \Psi\circ f\rangle(A),\end{aligned}$$

which together with (8.19) implies the assertion. □

As an immediate consequence of Theorem 8.17, we can prove the following theorem called *energy image density property* (wee [CF, Theorem 4.3.8] for the case $p = 2$), whose proof is essentially the same as in [Shi24, Proposition 7.6].

Corollary 8.18. *For any $f \in \mathcal{F}_p \cap \mathcal{C}(K)$, it holds that the image measure of $\Gamma_p\langle f\rangle$ by f is absolutely continuous with respect to the one-dimensional Lebesgue measure $\mathscr{L}^1$ on $\mathbb{R}$. In particular, $\Gamma_p\langle f\rangle(\{x\}) = 0$ for any $x \in K$.*

The same proof as [Shi24, Theorem 7.7] implies the following 'strong locality in a measure sense'.

Corollary 8.19. *Let $f, g \in \mathcal{F}_p \cap \mathcal{C}(K)$. If $(f-g)|_A$ is constant for some Borel set $A \in \mathcal{B}(K)$, then $\Gamma_p\langle f\rangle(A) = \Gamma_p\langle g\rangle(A)$.*

Remark 8.20. Theorem 8.17, Corollaries 8.18 and 8.19 are restricted to $f \in \mathcal{F}_p \cap \mathcal{C}(K)$ because of the possibility of $m \perp \Gamma_p\langle f\rangle$. Indeed, for canonical Dirichlet forms on many fractals, such a singularity is expected [Hin05, KM20].

Next we prove the following *Poincaré-type inequalities* in this setting.

Theorem 8.21. *There exist $C_{\mathrm{P}} > 0$ and $A_{\mathrm{P}} \ge 1$ such that for any $f \in \mathcal{F}_p$, $x \in K$ and $r > 0$,*

$$\int_{B_d(x,r)} \left|f(y) - f_{B_d(x,r)}\right|^p m(dy) \le C_{\mathrm{P}}\, r^{d_{\mathrm{w}}(p)} \int_{B_d(x,A_{\mathrm{P}}r)} d\Gamma_p\langle f\rangle. \tag{8.20}$$

In particular,

$$\int_{B_d(x,r)} \left|f(y) - f_{B_d(x,r)}\right| m(dy) \le C_{\mathrm{P}}^{1/p}\, r^{d_{\mathrm{w}}(p)/p} \left(\int_{B_d(x,A_{\mathrm{P}}r)} d\Gamma_p\langle f\rangle\right)^{1/p}. \tag{8.21}$$

Proof. We show that $\sigma(\mathcal{E}_p) \in (0,\infty)$, where

$$\sigma(\mathcal{E}_p) := \inf\left\{ \mathcal{E}_p(f) \;\middle|\; f \in \mathcal{F}_p \cap \mathcal{C}(K),\, f\big|_{\ell_{\mathrm{L}}} \equiv 0, \int_K f\,dm = 1 \right\}.$$

By Theorem 1.1(ii), (v), we have

$$\mathcal{F}_{p,\mathrm{ave}}(\mathrm{L}) := \left\{ f \in \mathcal{F}_p \cap \mathcal{C}(K) \;\middle|\; f|_{\ell_{\mathrm{L}}} \equiv 0, \int_K f\,dm = 1 \right\} \neq \emptyset,$$

and hence $\sigma(\mathcal{E}_p) < \infty$. Let $\overline{\mathcal{F}}_{p,\mathrm{ave}}(\mathrm{L})$ denote the closure of $\mathcal{F}_{p,\mathrm{ave}}(\mathrm{L})$ with respect to the norm $\|\cdot\|_{\mathcal{F}_p}$. Clearly, $\int_K v\,dm = 1$ for any $v \in \overline{\mathcal{F}}_{p,\mathrm{ave}}(\mathrm{L})$. For each $n \in \mathbb{N}$, let $v_n \in \mathcal{F}_{p,\mathrm{ave}}(\mathrm{L})$ satisfy $\mathcal{E}_p(v_n) \leq \inf_{f \in \mathcal{F}_{p,\mathrm{ave}}(\mathrm{L})} \mathcal{E}_p(f) + n^{-1}$, and define $h_n \in \mathcal{C}(K)$ by

$$h_n := \sum_{i \in \{3,4,5\}} (F_i)_* v_n.$$

By Theorem 1.1(viii), we have $h_n \in \mathcal{F}_p \cap \mathcal{C}(K)$ and

$$\mathcal{E}_p(h_n) = 3\rho(p)\mathcal{E}_p(v_n) \leq 3\rho(p)\bigl(\sigma(\mathcal{E}_p) + n^{-1}\bigr).$$

We also have from Theorem 1.1(vi) that

$$\|h_n\|_{L^p(m)}^p \lesssim \|h_n - (h_n)_K\|_{L^p(m)}^p + m(K) \leq C\mathcal{E}_p(h_n) + m(K).$$

From these estimates, $\{h_n\}_{n\in\mathbb{N}}$ turns out to be a bounded sequence in $\mathcal{F}_p$ and hence, by Theorem 1.1(i), we get a subsequence $\{h_{n_k}\}_{k\in\mathbb{N}}$ and $h_\infty \in \mathcal{F}_p$ so that h_{n_k} converges weakly to h_∞ in $\mathcal{F}_p$. Mazur's lemma yields a sequence $\widetilde{h}_j \in \mathrm{conv}\{h_{n_k}\}_{k\geq j}$ $(j \in \mathbb{N})$ such that $\widetilde{h}_j$ converges to h_∞ in $\mathcal{F}_p$, and we then have

$$\mathcal{E}_p(h_\infty) = \lim_{j\to\infty} \mathcal{E}_p(\widetilde{h}_j) \leq \limsup_{j\to\infty} 3\rho(p)\bigl(\sigma(\mathcal{E}_p) + j^{-1}\bigr) = 3\rho(p)\sigma(\mathcal{E}_p),$$

and $\int_K h_\infty\,dm = \lim_{j\to\infty}\int_K \widetilde{h}_j\,dm = 3/8$. If $\sigma(\mathcal{E}_p) = 0$, then h_∞ should be a constant function by Theorem 1.1(vi). Since $\widetilde{h}_j = 0$ on $\bigcup_{i\in S\setminus\{3,4,5\}} K_i$, we have $h_\infty \equiv 0$, which contradicts $\int_K h_\infty\,dm = 3/8$. Therefore $\sigma(\mathcal{E}_p) > 0$.

Next we show that for any $f \in \mathcal{F}_p$,

$$|f_{K_v} - f_{K_w}|^p \leq 2^{p/(p-1)}\sigma(\mathscr{E}_p)^{-1}\rho(p)^{-n}\left[\Gamma_p\langle f\rangle(K_v) + \Gamma_p\langle f\rangle(K_w)\right]. \tag{8.22}$$

By Theorems 1.1(ii) and 1.2(ii), it suffices to assume that $f \in \mathcal{F}_p \cap \mathcal{C}(K)$. By replacing f with $f\circ\Phi$ for some $\Phi \in D_4$, we may assume that $F_v^{-1}(K_v \cap K_w) = \ell_{\mathrm{L}}$, $F_w^{-1}(K_v\cap K_w) = \ell_{\mathrm{R}}$. Without loss of generality, we assume that $f_{K_v} - f_{K_w} \neq 0$. The function $h := f\circ F_v - (f\circ F_w)\circ S_2 \in \mathcal{F}_p \cap \mathcal{C}(K)$ satisfies $\int_K h\,dm = f_{K_v} - f_{K_w}$, $h|_{\ell_{\mathrm{L}}} \equiv 0$ and

$$\mathcal{E}_p(h) \leq \bigl(\mathcal{E}_p(f\circ F_v)^{1/p} + \mathcal{E}_p(f\circ F_w)^{1/p}\bigr)^p \leq 2^{p/(p-1)}\bigl(\mathcal{E}_p(f\circ F_v) + \mathcal{E}_p(f\circ F_w)\bigr). \tag{8.23}$$

By Lemma 8.14, we know that $\rho(p)^n\mathcal{E}_p(f\circ F_z) \leq \Gamma_p\langle f\rangle(K_z)$ for any $z \in W_n$. Hence (8.23) yields the desired inequality.

Finally, we prove (8.20). For $r \in (0,\infty)$, let $n = n(r) \in \mathbb{Z}_{\geq 0}$ be the smallest non-negative integer such that $r \geq a_*^{-n}$. Set $W(x,r) := W_n[B_d(x,r)]$ and $U(x,r) :=$

$\bigcup_{w\in W(x,r)} K_w$. Since $\mathrm{diam}(K_w,d) = a_*^{-n}$ for any $w \in W(x,r)$, we easily see that $U(x,r) \subseteq B_d(x, A_{\mathrm{P}} r)$, where $A_{\mathrm{P}} := 2$. Also, there exists N_1 which is independent of x,r such that $\#W(x,r) \le N_1$ by the metric doubling property of (K,d) (see Proposition 8.4(ii), (iii)). It is easy to see that

$$\begin{aligned}&\int_{U(x,r)} \left|f(y) - f_{U(x,r)}\right|^p m(dy) \\ &\le 2^{p-1} \sum_{w\in W(x,r)} m(K_w) \left(⨍_{K_w} |f(y) - f_{K_w}|^p m(dy) + \left|f_{K_w} - f_{U(x,r)}\right|^p \right). \end{aligned} \tag{8.24}$$

By Theorem 1.1(vi) for $f \circ F_w$ and Lemma 8.14, we have

$$⨍_{K_w} |f - f_{K_w}|^p \, dm \le C\mathcal{E}_p(f \circ F_w) \le C\rho(p)^{-n}\Gamma_p\langle f\rangle(K_w), \tag{8.25}$$

where C is the constant in Theorem 1.1(vi). Note that, by Proposition 8.4(i),

$$f_{K_w} - f_{U(x,r)} = \frac{1}{m(U(x,r))} \sum_{v\in W(x,r)} (f_{K_w} - f_{K_v}) m(K_v).$$

Hence, by choosing $w' \in W(x,r) \setminus \{w\}$ so that $f_{K_w} - f_{K_{w'}} = \max_{v\in W(x,r)} |f_{K_w} - f_{K_v}|$,

$$\left|f_{K_w} - f_{U(x,r)}\right| \le \left|f_{K_w} - f_{K_{w'}}\right|,$$

which together with Hölder's inequality and (8.22) yields that

$$\left|f_{K_w} - f_{U(x,r)}\right|^p \le 2^{p/(p-1)+1} N_1^{p-1} \sigma(\mathcal{E}_p)^{-1} \rho(p)^{-n} \sum_{v\in W(x,r)} \Gamma_p\langle f\rangle(K_v). \tag{8.26}$$

Since $m(K_w)\rho(p)^{-n} = a_*^{-n d_{\mathrm{w}}(p)} \le r^{d_{\mathrm{w}}(p)}$ for any $w \in W_n$, we obtain (8.20) by combining (8.24), (8.25) and (8.26). □

The next proposition obtains bounds on p-energy measure expressed using the underlying metric and measure. By using (8.20) instead of (6.34) in the proof of Lemma 7.2, we immediately achieve the following 'local behavior of p-energy in terms of (fractional) Korevaar–Schoen expression'.

Proposition 8.22. *There exists $C > 0$ such that for all Borel set U of K and $f \in \mathcal{F}_p$, we have*

$$\limsup_{r\downarrow 0} \int_U ⨍_{B_d(x,r)} \frac{|f(x) - f(y)|^p}{r^{d_{\mathrm{w}}(p)}} m(dy)m(dx) \le C\Gamma_p\langle f\rangle(\overline{U}), \tag{8.27}$$

and

$$\Gamma_p\langle f\rangle(U) \le C \lim_{\delta\downarrow 0} \liminf_{r\downarrow 0} \int_{U_\delta} ⨍_{B_d(x,r)} \frac{|f(x) - f(y)|^p}{r^{d_{\mathrm{w}}(p)}} m(dy)m(dx). \tag{8.28}$$

Proof. Let $U \subseteq K$, $\delta > 0$ and $f \in \mathcal{F}_p$. The same argument using a maximal r-net $N_r(\subseteq U)$ of U to get (7.3) yields

$$\int_U \fint_{B_d(x,r)} \frac{|f(x)-f(y)|^p}{r^{d_{\mathrm{w}}(p)}}\, m(dy)m(dx) \lesssim \sum_{y\in N_r} \Gamma_p\langle f\rangle\bigl(B_d(y, 2A_{\mathrm{P}} r)\bigr).$$

Since $\sum_{y\in N_r} \mathbb{1}_{B_d(y,2A_{\mathrm{P}}r)} \lesssim \mathbb{1}_{U_{2A_{\mathrm{P}}r}}$ by the metric doubling property, we obtain (8.27).

Next we show (8.28). By Lemma 7.3, we have

$$\limsup_{n\to\infty} \widetilde{\mathcal{E}}^{(n)}_{p,V_n(U)}(f) \le C_0 \liminf_{r\downarrow 0} \int_{U_\delta} \fint_{B_d(x,r)} \frac{|f(x)-f(y)|^p}{r^{d_{\mathrm{w}}(p)}}\, m(dy)m(dx), \tag{8.29}$$

where $C_0 > 0$ is independent of U, δ, f. Let $k \in \mathbb{N}$ be large enough so that $\bigcup_{w\in W_k[U]} K_w \subseteq U_\delta$. Then we see that

$$\begin{aligned}\Gamma_p\langle f\rangle(U) &\le \mathfrak{m}_p\langle f\rangle\left(\bigcup_{w\in W_k[U]} \Sigma_w\right) = \rho(p)^k \sum_{w\in W_k[U]} \mathcal{E}_p(f\circ F_w)\\ &\lesssim \rho(p)^k \sum_{w\in W_k[U]} \varliminf_{n\to\infty} \widetilde{\mathcal{E}}^{(n)}_p(f\circ F_w) = \sum_{w\in W_k[U]} \varliminf_{n\to\infty} \widetilde{\mathcal{E}}^{(n+k)}_{p,S^n(w)}(f).\end{aligned}$$

For $w \in W_k[U]$, we observe that $S^n(w) \subseteq V_{n+k}(U_\delta)$. Therefore,

$$\Gamma_p\langle f\rangle(U) \lesssim \liminf_{n\to\infty} \sum_{w\in W_k[U]} \widetilde{\mathcal{E}}^{(n+k)}_{p,S^n(w)}(f) \le \limsup_{n\to\infty} \widetilde{\mathcal{E}}^{(n)}_{p,V_n(U_\delta)}(f).$$

Combining with (8.29) for U_δ, we obtain (8.28). □

Remark 8.23. Once we have energy measures and the Poincaré inequality, minor modifications of the proof of [Mur24, Theorem 2.9] show the following result: for any uniform domain U of K in the sense of [Mur24, Definition 2.3] and $f \in \mathcal{F}_p$, we have $\Gamma_p\langle f\rangle(\partial U) = 0$. Note that $\mathrm{int}_K K = K \setminus \mathcal{V}_0$ is a uniform domain in this sense.

Finally, we finish the proof of Theorems 1.2 and 1.5.

Proof of Theorem 1.2. The existence of energy measures follows from the construction described at the beginning of this subsection, which in turn follows from Theorem 1.1. Properties (ii)-(iv) follow from Propositions 8.15. The assertions in (vi) follow from Theorem 8.17 and Corollary 8.19.

It remains to prove (i). The property $\Gamma_p\langle f\rangle(K) = \mathcal{E}_p(f)$ is immediate from the definition of $\Gamma_p\langle f\rangle$. In order to prove the second assertion, note that for any $w \in W_n, n \in \mathbb{N}, f \in \mathcal{F}_p$, by the self-similarity (iv),

$$\Gamma_p\langle f\rangle(K_w) = \rho(p)^n \sum_{u\in W_n; K_u\cap K_w\neq\emptyset} \Gamma_p\langle f\circ F_u\rangle(K_u\cap K_w). \tag{8.30}$$

If $u \neq w$ and $u, v \in W_n$, then $K_u \cap K_w \subset F_u(\mathcal{V}_0)$ which has energy measure zero by Remark 8.23 and the self-similarity (iv). Therefore $\Gamma_p\langle f\rangle(K_w) = \rho(p)^n \Gamma_p\langle f \circ F_w\rangle(K_w) = \rho(p)^n \mathcal{E}_p(f \circ F_w)$ for any $w \in W_n, n \in \mathbb{N}, f \in \mathcal{F}_p$. □

Proof of Theorem 1.5. The Poincaré inequality and capacity upper bounds follow from Theorem 8.21 and Proposition 6.21 respectively after verifying the assumptions using Theorem 8.3. □

Remark 8.24. One can see that the arguments in this section work with minor modifications for the generalized Sierpiński carpets (in the sense of Barlow and Bass [BB99]; see also [BBKT, Kaj10] for a correction of the original definition) embedded in $\mathbb{R}^2$. To go beyond the planar case, there are two obstacles that we have to overcome. The first one is the restriction on p such that $d_{\mathrm{f}} - d_{\mathrm{w}}(p) < 1$, which was always assumed in Sections 3-5. In higher-dimensional cases, there may exist $p \in (1, \infty)$ such that $d_{\mathrm{f}} - d_{\mathrm{w}}(p) \geq 1$, so we need completely new techniques/arguments to cover the all case $p \in (1, \infty)$ in Sections 3-5. This seems to be a very challenging problem (see also Problem 10.1). Even for p with $d_{\mathrm{f}} - d_{\mathrm{w}}(p) < 1$, one has to verify both U-PI$_p(\beta)$ and U-cap$_{p,\leq}(\beta)$ (with the same exponent β) for the approximating graphs, which is the second obstacle. As done in Proposition 8.8 for the Sierpiński carpet, one can obtain these conditions if one knows a good behavior of the discrete p-capacities as in (8.4) or (8.6). Let us emphasize that the argument in [BK13, Proof of Lemma 4.4] (see also Theorem 8.5 for the statement of this result) showing (8.4) (and (8.6)) uses the planarity in an essential way. Fortunately, a recent paper by Anttila and Eriksson-Bique [AEB24+b], which appeared after the submission of our paper, shows the *multiplicative inequality* corresponding to (8.4) for any $p \in (1, \infty)$ on a large class of fractal spaces having nice symmetries including the generalized Sierpiński carpet. A part of their results can be regarded as a generalization of [BB90, Mc02] (the case $p = 2$), and it seems to be enough to obtain U-PI$_p(\beta)$ and U-cap$_{p,\leq}(\beta)$ for the generalized Sierpiński carpets beyond the planar case as long as $d_{\mathrm{f}} - d_{\mathrm{w}}(p) < 1$ is satisfied. See also [AEB24+b, Section 9].

9 Comparison with the Loewner space: Theorem 1.7

In this section, we obtain partial results towards the attainment problem, namely the last main result Theorem 1.7.

9.1 Newton-Sobolev space $N^{1,p}$

We start by briefly recalling the theory of first-order Sobolev spaces on metric measure spaces based on the notion of *upper gradients*. A comprehensive account of this theory can be found in [HKST] (see also [BB, Hei]).

Hereafter, we let (X, θ, μ) be a metric measure space in the sense of [HKST], i.e., (X, θ) is a separable metric space and μ is a locally finite Borel-regular (outer) measure on X. In addition, we always assume that $\mu(O) > 0$ for any non-empty open set O.

We first recall the notion of *modulus of curve families*.

Definition 9.1 (Modulus of curve families)**.** Let $p \in (0, \infty)$ and let Γ be a subset of Γ_{rect}, where Γ_{rect} denotes the set of rectifiable curves in (X, θ). A non-negative Borel function $\rho \in \mathscr{B}_+(X)$ is said to be *admissible for* Γ if $\inf_{\gamma \in \Gamma} \int_\gamma \rho \, ds \geq 1$, where $\int_\gamma \rho \, ds$ is the usual curve integral (see [HKST, Section 5.1]). The p-modulus of Γ is defined as

$$\mathrm{Mod}_p(\Gamma) = \inf\bigl\{ \|\rho\|_{L^p(\mu)}^p \bigm| \rho \text{ is admissible for } \Gamma \bigr\}.$$

We shall say that a property of curves holds for Mod_p-*a.e. curve* if the p-modulus of the set of curves for which the property fails to holds is zero.

The corresponding properties to the discrete case in Lemma 2.3 are also true for p-modulus on (X, θ, μ) [HKST, Section 5.2]. The next notion of minimal p-weak upper gradient of a function u plays the role of '$|\nabla u|$'. The notion of weak upper gradients was introduced in [HK98], where it was called 'very weak gradients'.

Definition 9.2 (Upper gradients)**.** Let $p \in (0, \infty)$, $u \colon X \to \mathbb{R}$ and $g \in \mathscr{B}_+(X)$. (Here, both u and g is defined on every points of X.) The Borel function g is called a *p-weak upper gradient of u* if

$$|u(x) - u(y)| \leq \int_\gamma g \, ds \quad \text{for } \mathrm{Mod}_p\text{-a.e. } \gamma \in \Gamma_{\mathrm{rect}}, \tag{9.1}$$

where x, y are endpoints of γ. If (9.1) holds for every compact rectifiable curve, then g is called an *upper gradient of u*.

A p-weak upper gradient g of u is said to be a *minimal p-weak upper gradient* if it is p-integrable with respect to the measure μ and if $g \leq g'$ μ-a.e. in X whenever g' is a p-integrable p-weak upper gradient of u. Such the minimal p-weak upper gradient of u is denoted by g_u.

Remark 9.3. There are several other ways to define upper gradients, and these coincide with each other on any compact metric space equipped with a finite Borel measure. See [AGS13, Section 4 and Theorem 7.4] for details. Our definition of g_u is the same as that of the upper gradient denoted by $|\nabla u|_{S,p}$ in [AGS13].

If $\{g \mid g \text{ is a } p\text{-integrable upper gradient of } u\} \neq \emptyset$, then the existence and uniqueness (up to a μ-null set) of minimal p-weak upper gradient are established by the direct method in calculus of variations [HKST, Theorem 6.3.20 and Lemma 6.2.8]. We also recall that $\|g_u\|_{L^p(\mu)}^p$ is the smallest $L^p(X, \mu)$-norm among all p-integrable p-weak upper gradient of u. For other basic properties on upper gradients, we refer to [BB, Hei, HKST].

For a locally Lipschitz function $u \colon X \to \mathbb{R}$, we define its *lower pointwise Lipschitz constant function* $\operatorname{lip} u \colon X \to [0, \infty)$ as

$$\operatorname{lip} u(x) \coloneqq \liminf_{r \downarrow 0} \sup_{y \in B(x,r)} \frac{|u(y) - u(x)|}{r}, \tag{9.2}$$

which gives a typical example of upper gradients [HKST, Lemmas 6.2.5 and 6.2.6].

Proposition 9.4. *If* $u\colon X \to \mathbb{R}$ *is a locally Lipschitz function, then* $\operatorname{lip} u \in \mathscr{B}_+(X)$ *is an upper gradient of* u*.*

Now we can define the function spaces $\widetilde{N}^{1,p}$ and $N^{1,p}$, which are called *Newton-Sobolev spaces* and introduced in [Sha00]. Let $p \in [1,\infty)$ and let

$$\widetilde{N}^{1,p}(X,\theta,\mu) := \left\{ u\colon X \to [-\infty,\infty] \;\middle|\; \begin{array}{c} u \text{ is } p\text{-integrable with respect to } \mu \text{ and there} \\ \text{exists a } p\text{-integrable } p\text{-weak upper gradient } g \text{ of } u \end{array} \right\}, \tag{9.3}$$

which is clearly a vector space (over $\mathbb{R}$). We equip $\widetilde{N}^{1,p}(X,\theta,\mu)$ with the seminorm $\|\cdot\|_{N^{1,p}(X,\theta,\mu)}$ given by

$$\|u\|_{N^{1,p}(X,\theta,\mu)} = \|u\|_{L^p(\mu)} + \|g_u\|_{L^p(\mu)}. \tag{9.4}$$

To get a normed space, we next consider a quotient space of $\widetilde{N}^{1,p}(X,\theta,\mu)$.

Definition 9.5 (Newton-Sobolev space $N^{1,p}$)**.** Let $p \in [1,\infty)$. Two functions $f, g \in \widetilde{N}^{1,p}(X,\theta,\mu)$ are said to be equivalent, $f \sim_{N^{1,p}} g$ for short, if $\|f-g\|_{N^{1,p}(X,\theta,\mu)} = 0$. Let us denote the equivalence class of f by $[f]_{N^{1,p}}$. The *Newton–Sobolev space* $N^{1,p}(X,\theta,\mu)$ is defined as the quotient normed space $\widetilde{N}^{1,p}(X,\theta,\mu)/\sim_{N^{1,p}}$, whose quotient norm associated with the semi-norm defined in (9.4) is also denoted by $\|\cdot\|_{N^{1,p}(X,\theta,\mu)}$. We also use $\|\cdot\|_{N^{1,p}}$ or $\|\cdot\|_{N^{1,p}(\mu)}$ to denote $\|\cdot\|_{N^{1,p}(X,\theta,\mu)}$.

For any $p \in [1,\infty)$, $N^{1,p}(X,\theta,\mu)$ is a Banach space [HKST, Theorem 7.3.6].

Remark 9.6. If (K,d,m) is the Sierpiński carpet given in Definition 8.1, then [HKST, Proposition 7.1.33] implies that $N^{1,p}(K,d,m)$ is trivial, i.e., $N^{1,p}(K,d,m) = L^p(K,m)$. This triviality is due to the fact that $\operatorname{Mod}_p(\Gamma_{\mathrm{rect}}(K,d)) = 0$. Such triviality of 1-modulus is proved by [LP04] and one can find a proof for any $p \geq 1$ in [MT, Proposition 4.3.3].

We recall Poincaré inequalities based on the notion of upper gradient.

Definition 9.7. Let $p \in [1,\infty)$. The metric measure space (X,θ,μ) is said to satisfy the (p,p)*-Poincaré inequality* if there exist $C_{\mathrm{P}} \in (0,\infty), A_{\mathrm{P}} \in [1,\infty)$ such that for any $x \in X, r > 0, u \in \widetilde{N}^{1,p}(X,\theta,\mu)$ and for any p-weak upper gradient g of u, we have

$$\int_{B_\theta(x,r)} \left|u(y) - u_{B_\theta(x,r),\mu}\right|^p \mu(dy) \leq C_{\mathrm{P}}\, r^p \int_{B_\theta(x,A_{\mathrm{P}}r)} g^p \, d\mu, \tag{(p,p)-PI$^{\mathrm{ug}}$}$$

where $u_{B_\theta(x,r),\mu} = ⨍_{B_\theta(x,r)} u\,d\mu$. In addition, (X,θ,μ) is said to satisfy the $(1,p)$*-Poincaré inequality* (or p*-Poincaré inequality* for short) if for any $x \in X, r > 0, u \in \widetilde{N}^{1,p}(X,\theta,\mu)$ and for any p-weak upper gradient g of u, we have

$$⨍_{B_\theta(x,r)} \left|u(y) - u_{B_\theta(x,r),\mu}\right| \mu(dy) \leq C_{\mathrm{P}}\, r \left(⨍_{B_\theta(x,A_{\mathrm{P}}r)} g^p \, d\mu \right)^{1/p}. \tag{p-PI$^{\mathrm{ug}}$}$$

Here 'ug' stands for upper gradient to distinguish it from Poincaré inequality corresponding to energy measures as shown in Theorem 8.21 or Poincaré inequality on graphs as shown in Theorem 4.2. Note that, by Hölder's inequality, (p,p)-$\mathrm{PI}^{\mathrm{ug}}$ implies p-$\mathrm{PI}^{\mathrm{ug}}$ without changing the constant A_{P}.

9.2 Lipschitz partition of unity and localized energies

In this subsection, we provide analogue results of Proposition 8.22. We focus on an upper bound on the "energy measure" $g_f^p\,d\mu$ because we do not use lower bounds in this paper.

We work in the same settings as in the previous section, i.e., (X,θ,μ) is a separable metric space and μ is a locally finite Borel-regular (outer) measure on X which is positive on any non-empty open subset of X. In addition, we let $p\in(1,\infty)$ throughout this subsection.

The following Lipschitz partition of unity is a well-known tool to approximate arbitrary functions in $\widetilde{N}^{1,p}(X,\theta,\mu)$ with Lipschitz functions (see [HKST, pp. 104–105]).

Lemma 9.8. *Let (X,θ) be a doubling metric space. Let $\{x_i : i\in I\}$ be a maximal r-separated subset for some $r>0$. Then there exists $C_1>0$ depending only on the doubling constant of (X,θ) and a collection of C_1/r-Lipschitz functions $\varphi_i\colon X\to[0,1]$ such that $\sum_{i\in I}\varphi_i\equiv 1$ and $\operatorname{supp}[\varphi_i]\subset B_\theta(x_i,2r_i)$ for all $i\in I$.*

The next lemma provides an estimate for upper gradients of discrete convolutions.

Lemma 9.9. *Suppose that (X,θ,μ) is volume doubling. Let $\{x_i : i\in I\}$ be a maximal r-separated subset of (X,θ) and let $\{\varphi_i\}_{i\in I}$ denote a Lipschitz partition of unity satisfying the properties described in Lemma 9.8. For a μ-integrable function $u\colon X\to\mathbb{R}$, define $u_r\colon X\to\mathbb{R}$ as*

$$u_r(x) \coloneqq \sum_{i\in I} u_{B_\theta(x_i,r),\mu}\,\varphi_i(x), \quad \text{where } u_{B_\theta(x_i,r),\mu} = \fint_{B_\theta(x_i,r)} u\,d\mu \text{ for all } i\in I. \tag{9.5}$$

There exists $C>0$ depending only on the doubling constant of μ such that

$$\operatorname{lip} u_r(x) \le Cr^{-1}\fint_{B_\theta(x,4r)} \left|u(z)-u_{B_\theta(x,4r),\mu}\right|\mu(dz) \quad \text{for all } x\in X. \tag{9.6}$$

Proof. In this proof, we write $u_{B_\theta(x,r)} = u_{B_\theta(x,r),\mu}$ for simplicity. For any $x,y\in X$ with $\theta(x,y)<r$, we have $\varphi_i(x)\vee\varphi_i(y)\neq 0$ only if $\theta(x_i,x)<3r$ and therefore $B_\theta(x_i,r)\subset$

$B_\theta(x,4r)$ whenever $\varphi_i(x) \vee \varphi_i(y) \neq 0$. Hence for all $x, y \in X$ with $\theta(x,y) < r$, we have

$$\begin{aligned}
|u_r(x) - u_r(y)| &= \left|\sum_{i\in I} u_{B_\theta(x_i,r)}(\varphi_i(x) - \varphi_i(y))\right| = \left|\sum_{i\in I}\bigl(u_{B_\theta(x_i,r)} - u_{B_\theta(x,4r)}\bigr)(\varphi_i(x) - \varphi_i(y))\right| \\
&\leq \sum_{i\in I, \theta(x,x_i)<4r} \left|\bigl(u_{B_\theta(x_i,r)} - u_{B_\theta(x,4r)}\bigr)(\varphi_i(x) - \varphi_i(y))\right| \\
&\leq C_1 r^{-1}\theta(x,y) \sum_{i\in I,\theta(x,x_i)<4r} \fint_{B_\theta(x_i,r)} \left|u(z) - u_{B_\theta(x,4r)}\right| \mu(dz) \\
&\leq C_2 r^{-1}\theta(x,y) \fint_{B_\theta(x,4r)} \left|u(z) - u_{B_\theta(x,4r)}\right| \mu(dz).
\end{aligned}$$

In the second and third line, we used Lemma 9.8. In the last line, we used the fact that μ is a doubling measure and that the set of $\#\{i \in I \mid \theta(x_i, x) < 4r\}$ is bounded by a constant that depends only on the doubling constant of (X,θ). □

It is well-known that the p-energy of a function in $\widetilde{N}^{1,p}(X,\theta,\mu)$ is bounded from above by a Koreervaar–Schoen type energy. We say that a function $u\colon X \to \mathbb{R}$ belongs to the *Korevaar–Schoen–Sobolev space* $KS^{1,p}(X,\theta,\mu)$ if $u \in L^p(X,\mu)$ and

$$\limsup_{\varepsilon\downarrow 0} \int_X \varepsilon^{-p} \fint_{B_\theta(x,\varepsilon)} |u(y) - u(x)|^p \,\mu(dy)\,\mu(dx) < \infty.$$

It is know that $KS^{1,p}(X,\theta,\mu) = B^1_{p,\infty}(X,\theta,\mu)$; see [Bau24, Lemma 3.2].

In the following proposition, we control the L^p-norm of the minimal p-weak upper gradient on arbitrary sets using a Korevaar–Schoen type energy. The statement and its proof is a slight extension of that of [HKST, Theorem 10.4.3] which deals with the case $B = X$.

Proposition 9.10. *Let (X,θ,μ) be volume doubling. There exists $C > 0$ such that for all $u \in KS^{1,p}(X,\theta,\mu)$, there exists $\widetilde{u} \in \widetilde{N}^{1,p}(X,\theta,\mu)$ such that $\widetilde{u} = u$ μ-almost everywhere and such that its minimal p-weak upper gradient $g_{\widetilde{u}}$ satisfies, for any Borel set $B \subseteq X$,*

$$\int_B g_{\widetilde{u}}^p \,d\mu \leq C \limsup_{\varepsilon\downarrow 0} \int_B \varepsilon^{-p} \fint_{B_\theta(y,\varepsilon)} |u(y) - u(x)|^p \,\mu(dy)\,\mu(dx). \tag{9.7}$$

Proof. For each $n \in \mathbb{N}$, consider a maximal n^{-1}-separated subset of (X,θ) and the corresponding Lipschitz partition of unity as given in Lemma 9.8. Let $v_n := u_{n^{-1}}$ denote the function defined in (9.5). Then by [HKST, Proof of Theorem 10.4.3], we have $\lim_{n\to\infty} \int_X |v_n - u|^p \,d\mu = 0$ and, by Lemma 9.9 and Jensen's inequality, there exists $C_1 > 0$ depending only on p and the doubling constant of μ such that

$$\varlimsup_{n\to\infty} \int_A \operatorname{lip} v_n(x)^p \,\mu(dx) \leq C_1 \varlimsup_{\varepsilon\downarrow 0} \int_A \varepsilon^{-p} \fint_{B_\theta(x,\varepsilon)} |u(y) - u(x)|^p \,\mu(dy)\,\mu(dx) < \infty, \tag{9.8}$$

for any Borel set A of X. Hence $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $\widetilde{N}^{1,p}(X,\theta,\mu)$. Therefore by Mazur's lemma and [HKST, Proposition 7.3.7, Theorem 7.3.8], there exists $\widetilde{u} \in \widetilde{N}^{1,p}(X,\theta,\mu)$ such that $\widetilde{u} = u$ μ-a.e. and $g \in \mathscr{B}_+(X)$ satisfies the following properties. The function g is a p-weak upper gradient of $\widetilde{u}$ and is a limit in $L^p(X,\mu)$ of a sequence $\{g_j\}_{j\in\mathbb{N}}$ such that g_j is a convex combination of elements in the sequence $\{\operatorname{lip} v_j\}_{j\in\mathbb{N}}$ for all j and for any $n \in \mathbb{N}$ all but finitely many elements of g_j are finite convex combinations of $\operatorname{lip} v_j$ with $j \geq n$. Hence by Lemma 9.9,

$$\begin{aligned}\int_B g_{\widetilde{u}}^p \, d\mu \leq \int_B g^p \, d\mu &\leq \limsup_{n\to\infty} \int_B (\operatorname{lip} v_n)^p \, d\mu \\ &\overset{(9.8)}{\leq} C \limsup_{\varepsilon\downarrow 0} \int_B \varepsilon^{-p} ⨍_{B_\theta(y,\varepsilon)} |u(y)-u(x)|^p \, \mu(dy)\, \mu(dx),\end{aligned}$$

which completes the proof. □

9.3 Loewner metric and measure

Let us recall the definition of Loewner spaces.

Definition 9.11 (Loewner space; [CE, Definition 1.8]). Let $p \in (1,\infty)$ and let (X,θ,μ) be a metric measure space such that is metric doubling. The metric measure space (X,θ,μ) is said to be *p-Loewner* if μ is p-Ahlfors regular with respect to θ and p-Poincaré inequality p-$\mathrm{PI}^{\mathrm{ug}}$ holds. If (X,θ,μ) is p-Loewner for some $p \in (1,\infty)$, then θ is called a *Loewner metric* and μ is called a *Loewner measure*.

The original definition of *Loewner spaces* due to Heinonen and Koskela [HK98, Definition 3.1] is based on lower bounds on modulus. However, this gives an equivalent one by virtue of [HK98, Theorems 5.7 and 5.12]. This celebrated work identified Loewner spaces as the abstract setting where much of the nice properties of quasiconformal maps on Euclidean spaces are available.

The next result is an observation due to Cheeger and Eriksson-Bique [CE]. It states that, for a metric measure space satisfying the combinatorial Loewner property, any metric and measure attaining the Ahlfors regular conformal dimension yields a Loewner space. We recall this short argument as it plays a key role in rest of this section.

Proposition 9.12 ([CE, §1.6]). *Let (K,d,m) be the planar Sierpiński carpet in Definition 8.1. Suppose that the Ahlfors regular conformal dimension of (K,d,m) ($\dim_{\mathrm{ARC}}$ for short) is attained, i.e., there exists a metric $\theta \in \mathcal{J}(K,d)$ equipped with a $\dim_{\mathrm{ARC}}$-Ahlfors regular measure μ with respect to θ. Then (K,θ,μ) is a $\dim_{\mathrm{ARC}}$-Loewner space.*

Proof. This result follows from the $\dim_{\mathrm{ARC}}$-combinatorial Loewner property of PSC, which is proved in [BK13, Theorem 4.1]. As explained in [CE, §1.6], $\dim_{\mathrm{ARC}}$-combinatorial Loewner property along with $\dim_{\mathrm{ARC}}$-Ahlfors regularity implies $\dim_{\mathrm{ARC}}$-Loewner property in the sense of [HK98, (3.2)]. This is due to a result of Haïssinky [Haï09, Proposition B.2] comparing combinatorial and continuous versions of modulus and a different equivalent definition of the Loewner property in Heinonen and Koskela's celebrated work [HK98, Definition 3.1, Theorems 5.12 and 5.7]. $\square$

Recall from Definition 1.6 that the Ahlfors regular conformal dimension concerns the existence of a metric $\theta \in \mathcal{J}(X,d)$ and p-Ahlfors regular measure μ on (X,θ). It is well-known that the measures and metrics satisfying these conditions determine each other; that is μ can be recovered from θ and θ can be recovered from μ (up to a bounded multiplicative constant). We recall this fact in Lemmas 9.13 and 9.15.

Lemma 9.13. *Let $p \in (1,\infty)$ and let (X,θ,μ) be a metric measure space. If μ is p-Ahlfors regular with respect to θ, then there exists $C \geq 1$ (depending only on p and the doubling constant of θ) such that*

$$C^{-1}\mathscr{H}^p_\theta(B) \leq \mu(B) \leq C\mathscr{H}^p_\theta(B) \quad \text{for all Borel set } B \in \mathcal{B}(X), \tag{9.9}$$

where $\mathscr{H}^p_\theta$ denotes the p-dimensional Hausdorff measure with respect to the metric θ.

We also note that, by Lemma 9.13, the Ahlfors regularity can be regarded as a property on metrics (and the corresponding Hausdorff measures).

Conversely, David–Semmes deformation theory ([DS90] for example) allows us to construct a corresponding metric associated to a given Ahlfors regular measure μ that is bi-Lipschitz equivalent to the original Loewner metric. See also [Hei, Chapter 14] or [MT, Section 7.1]. To describe this we recall the definition of a maximal semi-metric.

Definition 9.14. A function $r : X \times X \to [0,\infty)$ is said to be a *semi-metric*, if it satisfies all the properties of a metric except possibly the property that $r(x,y) = 0$ implies $x = y$.

Let $h : X \times X \to [0,\infty)$ be an arbitrary function. Then there exists a unique maximal semi-metric $d_h : X \times X \to [0,\infty)$ such that $d_h(x,y) \leq h(x,y)$ for all $x,y \in X$ [BBI, Lemma 3.1.23]. We say that d_h is the *maximal semi-metric induced by h*. More concretely, d_h can be defined as follows. Let $\widetilde{h}(x,y) = \min(h(x,y),h(y,x))$. Then

$$d_h(x,y) = \inf\left\{\sum_{i=0}^{N-1}\widetilde{h}(x_i,x_{i+1}) : N \in \mathbb{N}, x_0 = x, x_N = y\right\}. \tag{9.10}$$

To following lemma follows easily from the definitions.

Lemma 9.15. *Let $p \in (1,\infty)$ and let (X,d) be a metric space. If $\theta \in \mathcal{J}(X,d)$ and μ is a Borel measure on X such that μ is p-Ahlfors regular with respect to θ. Let $h(x,y) := \mu(B_d(x,d(x,y)))^{1/p}$ for all $x,y \in X$ and let d_h denote the maximal semi-metric. Then d_h is bi-Lipschitz equivalent to θ, that is, there exists $C > 1$ such that*

$$C^{-1}\theta(x,y) \leq d_h(x,y) \leq C\theta(x,y) \quad \text{for all } x,y \in X.$$

In particular $d_h \in \mathcal{J}(X,d)$ and μ is p-Ahlfors regular on (X,d_h).

The key point of the above Lemma is that the definition of d_h depends only on the measure μ and d and not on θ. Nevertheless, as the conclusion shows d_h is bi-Lipschitz equivalent to θ. The proof relies on the observation that μ being a doubling measure and $\theta \in \mathcal{J}(X,d)$ implies that $\mu(B_d(x,d(x,y)))$ is comparable to $\mu(B_\theta(x,\theta(x,y)))$.

In the rest of this paper, we discuss the structures of metrics and measures that attain the Ahlfors regular conformal dimension of the Sierpiński carpet if exist. In view of Lemma 9.15, we focus on optimal measures. We introduce the standing framework in the remaining part:

Assumption 9.16. Let (K,d,m) be the planar Sierpiński carpet in Definition 8.1. Let $d_{\mathrm{f}} = \log 8/\log 3$ and $p = \dim_{\mathrm{ARC}}(K,d,m)$. We suppose the attainment of $\dim_{\mathrm{ARC}}(K,d,m)$. Let $\theta \in \mathcal{J}(K,d)$ and let μ be a Borel-regular measure on K such that μ is p-Ahlfors regular with respect to θ.

Remark 9.17. By the results of [KL04, Tys00] (see also [MT, Section 4.3] for a review of related results), we know that

$$1 < 1 + \frac{\log 2}{\log 3} \le p = \dim_{\mathrm{ARC}}(K,d,m) < d_{\mathrm{f}}. \tag{9.11}$$

By [BK13, Corollary 3.7 and Theorem 4.1] or alternately by [Kig20, Theorem 4.7.6], we have $d_{\mathrm{w}}(p) = d_{\mathrm{f}}$.

B. Kleiner [Kle] observed than any optimal measure μ is mutually singular to the self-similar measure m. Although we don't need this fact, it helps us to elucidate that the comparison of norms on Theorem 1.7(i) does not follow comparison of corresponding semi-norms as the $L^p(m)$ and $L^p(\mu)$ norms are not comparable.

Proposition 9.18 (due to Bruce Kleiner)**.** *Under Assumption 9.16, the measures m and μ are mutually singular.*

Proof. This proof by contradiction uses a 'blow-up' argument. Assume to the contrary that μ is not singular to m. Let $\mu = \mu_a + \mu_s$ denote the Lebesgue decomposition of μ with respect to m, where $\mu_a \ll m$, $\mu_s \perp m$ and $\mu_a \neq 0$ by assumption. Let $f = \frac{d\mu_a}{dm}$. For m-almost every $x \in K$, we have ([KM20, Proposition A.4])

$$\lim_{r\downarrow 0} \frac{\mu_s(B_d(x,r))}{m(B_d(x,r))} = 0 \tag{9.12}$$

and for m-almost every $x \in \{y \in K : f(y) > 0\}$, we have ([Hei, (2.8)])

$$\lim_{r\downarrow 0} \frac{1}{m(B_d(x,r))} \int_{B_d(x,r)} |f(y) - f(x)|\, m(dy) = 0. \tag{9.13}$$

Since $\mu_a \neq 0$, there exists $x \in \{y \in K : f(y) > 0\}$ such that both (9.12) and (9.13) hold. Pick $\omega \in \Sigma$ such that $\chi(\omega) = x$ and set $w_n := [\omega]_n \in W_n$ for all $n \in \mathbb{N}$. Define a sequence of probability measures μ_n and metrics $\theta_n \colon K \times K \to [0,\infty)$ as

$$\mu_n(A) := \frac{\mu(F_{w_n}(A))}{\mu(K_{w_n})}, \quad \theta_n(x,y) := \frac{\theta(F_{w_n}(x), F_{w_n}(y))}{\operatorname{diam}(K_{w_n},\theta)}, \quad \text{for all } n \in \mathbb{N},$$

where $\theta \in \mathcal{J}(K,d)$ is such that μ is p-Ahlfors regular in (K,θ) and p is as given in Assumption 9.16. By (9.12) and (9.13), the sequence of measures μ_n converges to $f(x)m$ in the topology of weak convergence. Furthermore, it is easy to verify that there exists a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that the identity map $\mathrm{Id} : (K,\theta_n) \to (K,d)$ is an η-quasisymmetry for all $n \in \mathbb{N}$. By the same argument as [KM23, Proof of Proposition 6.18] using Arzela–Ascoli theorem, there exists a subsequence $\{\theta_{n_k}\}_{k\in\mathbb{N}}$ of $\{\theta_n\}_{n\in\mathbb{N}}$ converging uniformly to $\widetilde{\theta} \in C(K \times K)$. This along with $\mathrm{diam}(K,\theta_n) = 1$ implies that $\widetilde{\theta}$ is a metric on K, $\mathrm{Id} : (K,\widetilde{\theta}) \to (K,d)$ is a η-quasisymmetry and hence $\widetilde{\theta} \in \mathcal{J}(K,d)$. This implies that the measure $f(x)m$ is p-Ahlfors regular in $(K,\widetilde{\theta})$. Therefore by Lemma 9.15, we obtain $p = d_{\mathrm{f}}$ which contradicts (9.11). □

In the next subsection, we compare (K,d,m) and (K,θ,μ). To state a relation between m and μ, we will use the following notion called *minimal energy-dominant measures*.

Definition 9.19. Let $(\mathcal{E}_p,\mathcal{F}_p)$ be as given in Theorem 1.1. A Borel-regular finite measure ν on K is called a *minimal energy-dominant measure of* $(\mathcal{E}_p,\mathcal{F}_p)$ if the following conditions (i) and (ii) hold.

(i) (Domination) For every $f \in \mathcal{F}_p$, we have $\Gamma_p\langle f\rangle \ll \nu$.

(ii) (Minimality) For another Borel-regular finite measure ν' satisfying the above 'domination' property, we have $\nu \ll \nu'$.

In Dirichlet form theory, the existence of such a measure is shown in [Nak85, Lemma 2.2]. The following lemma gives basic results on minimal energy-dominant measures of $(\mathcal{E}_p,\mathcal{F}_p)$, whose proofs are straightforward modifications of [Hin10, Lemmas 2.2-2.4]. (Lemma 9.20 holds in a more general setting; indeed, its proof relies only on the triangle inequality for p-energy measures, the fact that $\sup_{A\in\mathcal{B}(K)}\Gamma_p\langle f\rangle(A) \le \mathcal{E}_p(f)$ for any $f \in \mathcal{F}_p$ and the separability of $\mathcal{F}_p$.)

Lemma 9.20. *Let* $(\mathcal{E}_p,\mathcal{F}_p)$ *be as given in Theorem 1.1.*

(a) *Let ν be a Borel-regular finite measure on K and let $f, f_n \in \mathcal{F}_p$ $(n \in \mathbb{N})$ such that $\mathcal{E}_p(f - f_n) \to 0$ as $n \to \infty$. Suppose that $\Gamma_p\langle f_n\rangle \ll \nu$ for all $n \in \mathbb{N}$. Then $\Gamma_p\langle f\rangle \ll \nu$.*

(b) *Let $\{f_n\}_{n\in\mathbb{N}}$ be a dense subset of $\mathcal{F}_p$ (recall Theorem 1.1(i)). Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n\mathcal{E}_p(f_n)$ converges. Then $\nu := \sum_{n=1}^{\infty} a_n\Gamma_p\langle f_n\rangle$ defines a minimal energy-dominant measure of $(\mathcal{E}_p,\mathcal{F}_p)$.*

(c) *Let ν be a minimal energy-dominant measure of $(\mathcal{E}_p,\mathcal{F}_p)$ and let $A \in \mathcal{B}(K)$. Then $\nu(A) = 0$ if and only if $\Gamma_p\langle f\rangle(A) = 0$ for all $f \in \mathcal{F}_p$.*

Proof. (a): For any $A \in \mathcal{B}(K)$ satisfying $\nu(A) = 0$, by Theorem 1.2(ii),

$$\Gamma_p\langle f\rangle(A)^{1/p} \le \Gamma_p\langle f_n\rangle(A)^{1/p} + \Gamma_p\langle f - f_n\rangle(A)^{1/p} \le \mathcal{E}_p(f - f_n)^{1/p} \xrightarrow[n\to\infty]{} 0,$$

whence it follows that $\Gamma_p\langle f\rangle \ll \nu$.

(b): By the definition of ν, we note that $\Gamma_p\langle f_n\rangle(A) = 0$ for any $n \in \mathbb{N}$ whenever $A \in \mathcal{B}(K)$ satisfies $\nu(A) = 0$, in particular, $\Gamma_p\langle f\rangle \ll \nu$ for any $f \in \mathcal{F}_p$ by (a). Hence it suffices to show the minimality of ν. Let ν' be another Borel-regular finite measure on K such that $\Gamma_p\langle f\rangle \ll \nu'$ for any $f \in \mathcal{F}_p$. Then for any $A \in \mathcal{B}(K)$ satisfying $\nu'(A) = 0$, we have $\Gamma_p\langle f_n\rangle(A) = 0$ for any $n \in \mathbb{N}$ and thus $\nu(A) = 0$. This proves $\nu \ll \nu'$.

(c): It is clear that, for $A \in \mathcal{B}(K)$, $\nu(A) = 0$ implies $\Gamma_p\langle f\rangle(A) = 0$ for any $f \in \mathcal{F}_p$. To prove the converse, assume that $A \in \mathcal{B}(K)$ satisfies $\Gamma_p\langle f\rangle(A) = 0$ for any $f \in \mathcal{F}_p$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a dense subset of $\mathcal{F}_p$ and let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n\mathcal{E}_p(f_n)$ converges. (For example, $a_n = 2^{-n}\bigl(\mathcal{E}_p(f_n)^{-1} \wedge 1\bigr)$.) Then $\nu' := \sum_{n=1}^{\infty} a_n\Gamma_p\langle f_n\rangle$ is also a minimal energy-dominant measure of $(\mathcal{E}_p, \mathcal{F}_p)$ by (b), so the minimality conditions of ν and of ν' imply that ν and ν' are mutually absolutely continuous. Since $\nu'(A) = 0$ by the assumption that $\Gamma_p\langle f\rangle(A) = 0$ for any $f \in \mathcal{F}_p$, we obtain $\nu(A) = 0$. □

9.4 Identifying self-similar and Newtonian Sobolev spaces

In this subsection, we will compare different notions of energies ($\mathcal{E}_p(f)$ and $\int_K g_f^p\,d\mu$) and Sobolev spaces ($\mathcal{F}_p$ and $N^{1,p}$) on the Sierpiński carpet under assuming the attainment of its Ahlfors regular conformal dimension. Throughout this subsection, we always suppose Assumption 9.16.

The following is a two-weight Poincaré type inequality, which is the key ingredient to compare the two different geometries (self-similar and Loewner).

Proposition 9.21. *Suppose Assumption 9.16. There exist $C, A > 1$ such that for all $x \in K, r > 0$, we have*

$$\inf_{\alpha\in\mathbb{R}} \int_{B_d(x,r)} |f-\alpha|^p\,dm \le Cr^{d_f}\int_{B_d(x,Ar)} g_f^p\,d\mu \quad \text{for all } f \in N^{1,p}(K,\theta,\mu)\cap\mathcal{C}(K), \tag{9.14}$$

$$\inf_{\alpha\in\mathbb{R}} \int_{B_\theta(x,r)} |f-\alpha|^p\,d\mu \le Cr^{p}\,\Gamma_p\langle f\rangle(B_\theta(x,Ar)) \quad \text{for all } f \in \mathcal{F}_p(K,d,m)\cap\mathcal{C}(K), \tag{9.15}$$

where g_f is the minimal p-weak upper gradient of f.

Proof. In this proof, each function in $N^{1,p}(K,\theta,\mu)\cap\mathcal{C}(K)$ (or $\mathcal{F}_p(K,d,m)\cap\mathcal{C}(K)$) is considered as a pointwisely defined continuous function on K. Fix $p_1 \in (p,\infty)$. To prove (9.14), by [Hei, Lemma 4.22] and d_f-Ahlfors regularity of (K,d,m), it suffices to show the following weak type estimate: There exist $C_1, A_1 \in (1,\infty)$ such that

$$\inf_{\alpha\in\mathbb{R}} \sup_{t>0} t^{p_1} m\left(\{y \in B_d(x,r) : |f(y)-\alpha| > t\}\right) \le C_1 r^{d_f}\left(\int_{B_d(x,A_1r)} g_f^p\,d\mu\right)^{p_1/p} \tag{9.16}$$

for all $f \in N^{1,p}(K,\theta,\mu)\cap\mathcal{C}(K)$.

Note that by Proposition 9.12, the space (X,θ,μ) satisfies the Poincare inequality p-PI$^{\mathrm{ug}}$. Let $A_{\mathrm{P}} \in [1,\infty)$ denote the constant in p-PI$^{\mathrm{ug}}$ as given in Definition 9.7. Since $\theta \in \mathcal{J}(K,d)$, by [MT, Lemma 1.2.18], there exists $A \in (1,\infty)$ such that for all $x \in K, r > 0$, there exists $s \in (0, \mathrm{diam}(K,\theta)]$ satisfying

$$B_d(x,r) \subset B_\theta(x,s) \subset B_\theta(x,(1+2A_{\mathrm{P}})s) \subset B_d(x,Ar). \tag{9.17}$$

By p-PI$^{\mathrm{ug}}$ and p-Ahlfors regularity of (K,θ,μ), there exists $C_2 > 1$ such that for all $x \in K, s > 0, y \in B_\theta(x,s), f \in \widetilde{N}^{1,p}(K,\theta,\mu)$, we have

$$\begin{aligned}\left| ⨍_{B_\theta(y,s)} f\,d\mu - ⨍_{B_\theta(x,2s)} f\,d\mu \right| &\le \frac{1}{\mu(B_\theta(y,s))} \int_{B_\theta(x,2s)} \left| f - ⨍_{B_\theta(x,2s)} f\,d\mu \right| d\mu \\ &\le C_2 \left(\int_{B_\theta(x,2A_{\mathrm{P}}s)} g_f^p\,d\mu \right)^{1/p}. \end{aligned}\tag{9.18}$$

By a similar argument, there exists $C_3 > 1$ such that for all $x \in K, s > 0, y \in B_\theta(x,s), i \in \mathbb{Z}_{\ge 0}, f \in \widetilde{N}^{1,p}(K,\theta,\mu)$, we have

$$\left| ⨍_{B_\theta(y,2^{-i}s)} f\,d\mu - ⨍_{B_\theta(y,2^{-i-1}s)} f\,d\mu \right| \le C_3 \left(\int_{B_\theta(y,A_{\mathrm{P}}2^{-i}s)} g_f^p\,d\mu \right)^{1/p}. \tag{9.19}$$

Note that (K,θ) is connected since (K,θ) is homeomorphic to (K,d). By the reverse doubling property [Hei, Exercise 13.1] of m with respect to the metric θ, there exists $c_4 \in (0,1)$ such that for all $y \in K, s \in (0,\mathrm{diam}(K,\theta)]$, we have

$$c_4 \sum_{i=0}^{\infty} \left(\frac{m(B_\theta(y,2^{-i}s))}{m(B_\theta(y,s))} \right)^{1/p_1} < \frac{1}{2}. \tag{9.20}$$

In order to show (9.16), for any $f \in N^{1,p}(K,\theta,\mu) \cap \mathcal{C}(K)$, we choose $\alpha = ⨍_{B_\theta(x,2s)} f\,d\mu$. If $t \le 2C_2 \left(\int_{B_d(x,Ar)} g_f^p\,d\mu \right)^{1/p}$, the estimate (9.16) follows from the d_{f}-Ahlfors regularity of (K,d,m). Therefore, it suffices to consider the case $t > 2C_2 \left(\int_{B_d(x,Ar)} g_f^p\,d\mu \right)^{1/p}$. By (9.17), (9.18), we have

$$\{y \in B_d(x,r) : |f(y) - \alpha| > t\} \subset \left\{ y \in B_d(x,r) : \left| f(y) - ⨍_{B_\theta(y,s)} f\,d\mu \right| > t/2 \right\} \tag{9.21}$$

for all $t > 2C_2 \left(\int_{B_d(x,Ar)} g_f^p\,d\mu \right)^{1/p}$. By (9.21), for any $y \in B_d(x,r)$ such that

$$\left| f(y) - ⨍_{B_\theta(x,2s)} f\,d\mu \right| > t > 2C_2 \left(\int_{B_d(x,Ar)} g_f^p\,d\mu \right)^{1/p},$$

we have

$$\begin{aligned} c_4 \sum_{i=0}^{\infty} \left(\frac{m(B_\theta(y, 5A_{\mathrm{P}} 2^{-i} s))}{m(B_\theta(y, 5A_{\mathrm{P}} s))} \right)^{1/p_1} t < t/2 \qquad &\text{(by (9.20))} \\ < \left| f(y) - \fint_{B_\theta(y,s)} f \, d\mu \right| \qquad &\text{(by (9.21))} \\ \leq C_3 \sum_{i=0}^{\infty} \left(\int_{B_\theta(y, A_{\mathrm{P}} 2^{-i} s)} g_f^p \, d\mu \right)^{1/p} \qquad &\text{(by (9.19))}. \end{aligned}$$

Therefore there exists $C_5 > 1$ such that following property holds: For each $y \in B_d(x,r)$ that satisfies $\left| f(y) - \fint_{B_\theta(x,2s)} f \, d\mu \right| > t > 2C_2 \left(\int_{B_d(x,Ar)} g_f^p \, d\mu \right)^{1/p}$, there exists $i_y \in \mathbb{Z}_{\geq 0}$ such that

$$m(B_\theta(y, 5A_{\mathrm{P}} 2^{-i_y} s)) \leq C_5 t^{-p_1} r^{d_{\mathrm{f}}} \left(\int_{B_\theta(y, A_{\mathrm{P}} 2^{-i_y} s)} g_f^p \, d\mu \right)^{p_1/p}. \tag{9.22}$$

By the $5B$ covering lemma [Hei, Theorem 1.2], there exists a pairwise disjoint collection of balls $\{B_\theta(y_j, A_{\mathrm{P}} 2^{-i_{y_j}} s) \mid j \in J\}$ with $y_j \in B_d(x,r)$ for all $j \in J$ such that

$$\left\{ y \in B_d(x,r) : \left| f(y) - \fint_{B_\theta(x,2s)} f \, d\mu \right| > t \right\} \subseteq \bigcup_{j \in J} B_\theta(y_j, 5A_{\mathrm{P}} 2^{-i_{y_j}} s).$$

Hence

$$\begin{aligned} & m\left(\left\{ y \in B_d(x,r) : \left| f(y) - \fint_{B_\theta(x,2s)} f \, d\mu \right| > t \right\} \right) \\ & \leq \sum_{j \in J} m\left(B_\theta(y_j, 5A_{\mathrm{P}} 2^{-i_{y_j}} s) \right) \\ & \overset{(9.22)}{\leq} C_5 t^{-p_1} r^{d_{\mathrm{f}}} \sum_{j \in J} \left(\int_{B_\theta(y_j, A_{\mathrm{P}} 2^{-i_{y_j}} s)} g_f^p \, d\mu \right)^{p_1/p} \\ & \leq C_5 t^{-p_1} r^{d_{\mathrm{f}}} \left(\sum_{j \in J} \int_{B_\theta(y_j, A_{\mathrm{P}} 2^{-i_{y_j}} s)} g_f^p \, d\mu \right)^{p_1/p} \qquad \text{(since } p_1 > p\text{)} \\ & \leq C_5 t^{-p_1} r^{d_{\mathrm{f}}} \left(\int_{B_\theta(x, (1+A_{\mathrm{P}})s)} g_f^p \, d\mu \right)^{p_1/p} \\ & \overset{(9.17)}{\leq} C_5 t^{-p_1} r^{d_{\mathrm{f}}} \left(\int_{B_d(x,Ar)} g_f^p \, d\mu \right)^{p_1/p}, \end{aligned}$$

which concludes the proof of (9.16) and therefore (9.14).

The proof of (9.15) follows from a similar argument where the application of p-$\mathrm{PI}^{\mathrm{ug}}$ in (K, θ, μ) is replaced with (8.21) with $\beta = d_{\mathrm{w}}(p) = d_{\mathrm{f}}$ (see also Remark 9.17), which is the $(1,p)$-Poincaré inequality for the self-similar energy on (K, d, m). □

The following result compares energy measures and energies in the Sobolev spaces.

Theorem 9.22. *Suppose Assumption 9.16. Then we have*

$$\mathcal{F}_p(K,d,m)\cap\mathcal{C}(K)=N^{1,p}(K,\theta,\mu)\cap\mathcal{C}(K).$$

We let $\mathscr{C}_p:=\mathcal{F}_p(K,d,m)\cap\mathcal{C}(K)$*. In addition, there exists* $C>1$ *such that for any Borel set* $B\in\mathcal{B}(K)$ *and for all* $f\in\mathscr{C}_p$*, we have*

$$C^{-1}\Gamma_p\langle f\rangle(B)\leq\int_B g_f^p\,d\mu\leq C\Gamma_p\langle f\rangle(B), \tag{9.23}$$

where g_f *denotes the minimal p-weak upper gradient of* f*. In particular,*

$$C^{-1}\mathcal{E}_p(f)\leq\int_K g_f^p\,d\mu\leq C\mathcal{E}_p(f)\quad\text{for all } f\in\mathscr{C}_p. \tag{9.24}$$

Furthermore, there exists $C_1>0$ *such that*

$$C_1^{-1}\left\|f\right\|_{N^{1,p}(\mu)}\leq\left\|f\right\|_{\mathcal{F}_p}\leq C_1\left\|f\right\|_{N^{1,p}(\mu)}\quad\text{for all } f\in\mathscr{C}_p. \tag{9.25}$$

We start with a simpler condition to obtain comparability of measures whose proof is in Appendix A.

Lemma 9.23. *Let* (X,d) *be a doubling metric space. Let* ν_1,ν_2 *be two finite Borel measures on* X *satisfying the following property: There exist* $C_1\in(0,\infty),A_1\in(1,\infty)$ *such that for all* $x\in X,r>0$*, we have*

$$\nu_1(B_{\mathsf{d}}(x,r))\leq C_1\nu_2(B_{\mathsf{d}}(x,A_1r)).$$

Then there exists $C_2>0$ *such that*

$$\nu_1(B)\leq C_2\,\nu_2(B) \tag{9.26}$$

for all Borel set $B\subset X$*.*

Next we compare energy measures on balls for the spaces $N^{1,p}(K,\theta,\mu)$ and $\mathcal{F}_p(K,d,m)$.

Lemma 9.24. *Suppose Assumption 9.16. Then the following are true:*

(i) *We have* $\mathcal{F}_p(K,d,m)\cap\mathcal{C}(K)\subseteq N^{1,p}(K,\theta,\mu)\cap\mathcal{C}(K)$*. Moreover, there exist* $C>0,A>1$ *such that for all* $f\in\mathcal{F}_p(K,d,m)\cap\mathcal{C}(K),x\in K,r>0$*, we have*

$$\int_{B_\theta(x,r)}g_f^p\,d\mu\leq C\Gamma_p\langle f\rangle(B_\theta(x,Ar)). \tag{9.27}$$

(ii) *We have* $N^{1,p}(K,\theta,\mu)\cap\mathcal{C}(K)\subseteq\mathcal{F}_p(K,d,m)\cap\mathcal{C}(K)$*. Moreover, there exist* $C>0,A>1$ *such that for all* $f\in\mathcal{F}_p(K,d,m)\cap\mathcal{C}(K),x\in K,r>0$*, we have*

$$\Gamma_p\langle f\rangle(B_d(x,r))\leq C\int_{B_d(x,Ar)}g_f^p\,d\mu. \tag{9.28}$$

Proof. (i) We will start with the proof of (9.27). To this end, let $f \in \mathcal{F}_p(K,d,m) \cap \mathcal{C}(K), x \in K, r > 0$ be arbitrary. For $0 < s < r$, consider a maximal s-separated subset N of $B_\theta(x,r)$ in (K,θ), so that $B_\theta(x,r) \subseteq \cup_{y\in N} B_\theta(y,s) \subseteq B_\theta(x,r+s)$. Therefore

$$\mathbb{1}_{B_\theta(x,r)}(y)\mathbb{1}_{B_\theta(y,s)}(z) \le \sum_{n\in N} \mathbb{1}_{B_\theta(n,2s)}(y)\mathbb{1}_{B_\theta(n,2s)}(z). \tag{9.29}$$

By the doubling property and [HKST, Lemma 4.1.12], for any $\lambda > 1$, there exists C_λ depending only on λ and the doubling constant of (K,θ) such that

$$\sum_{n\in N} \mathbb{1}_{B_\theta(n,\lambda s)} \le C_\lambda \mathbb{1}_{B_\theta(x,r+\lambda s)}. \tag{9.30}$$

We will use Proposition 9.10 to show estimate the norm of the upper gradient. By (9.15) in Proposition 9.21, there exist $C_1, A_1 \in (1,\infty)$ such that for all $f \in \mathcal{F}_p(K,d,m) \cap \mathcal{C}(K)$, we have

$$\begin{aligned}
&\int_{B_\theta(x,r)} s^{-p} \fint_{B_\theta(y,s)} |f(y)-f(z)|^p \,\mu(dy)\,\mu(dz) \\
&\quad\lesssim s^{-2p} \int_K \int_K |f(y)-f(z)|^p \mathbb{1}_{B_\theta(x,r)}(y)\mathbb{1}_{B_\theta(y,s)}(z)\,\mu(dy)\,\mu(dz) \\
&\quad\lesssim s^{-2p} \sum_{n\in N} \int_{B_\theta(n,2s)} \int_{B_\theta(n,2s)} |f(y)-f(z)|^p \,\mu(dy)\,\mu(dz) \quad (\text{by } (9.29)) \\
&\quad\lesssim s^{-2p} \sum_{n\in N} \int_{B_\theta(n,2s)} \int_{B_\theta(n,2s)} \left(\left|f(y) - f_{B_\theta(n,2s),\mu}\right|^p + \left|f(z) - f_{B_\theta(n,2s),\mu}\right|^p \right) \mu(dy)\,\mu(dz) \\
&\quad\lesssim s^{-p} \sum_{n\in N} \inf_{\alpha\in\mathbb{R}} \int_{B_\theta(n,2s)} |f(y)-\alpha|^p \,\mu(dy) \quad (\text{by [BB, Lemma 4.17] and } \mathrm{AR}(p) \text{ for } \mu) \\
&\quad\lesssim \sum_{n\in N} \Gamma_p\langle f\rangle(B_\theta(n, A_1 s)) \quad (\text{by } (9.15)) \\
&\quad\le C_1 \Gamma_p\langle f\rangle(B_\theta(x, r + A_1 s)) \quad (\text{by } (9.30)).
\end{aligned} \tag{9.31}$$

By letting $r \to \infty$ in (9.31) and using Proposition 9.10, we conclude that

$$\mathcal{F}_p(K,d,m) \cap \mathcal{C}(K) \subseteq N^{1,p}(K,\theta,\mu) \cap \mathcal{C}(K).$$

By (9.31) and (9.7) in Proposition 9.10, we obtain (9.27).

(ii) This is similar to part (i), except that we use Proposition 8.22, $\mathrm{AR}(d_{\mathrm{f}})$ for m and (9.14) in place of Proposition 9.10, $\mathrm{AR}(p)$ for μ and (9.15) respectively. □

Proof of Theorem 9.22. The estimate (9.23) follows from Lemma 9.24 along with Lemma 9.23.

It remains to show (9.25). By normalizing the measures if necessary, we assume that m and μ are probability measures. For $f \in \mathcal{C}(K)$, let $f_m = \int_K f\,dm$ and $f_\mu = \int_K f\,d\mu$

denote the averages of f with respect to m and μ respectively. The proof of (9.15) with $r = 2\operatorname{diam}(K,\theta)$ yields

$$\int_K |f - f_m|^p \, d\mu \lesssim \|f\|_{\mathcal{F}_p}^p \quad \text{for all } f \in \mathcal{F}_p(K,d,m) \cap \mathcal{C}(K). \tag{9.32}$$

Note that for any $f \in \mathcal{F}_p(K,d,m) \cap \mathcal{C}(K)$, we have

$$\begin{aligned}\int_K |f|^p \, d\mu &\le 2^{p-1} \left(|f_m|^p + \int_K |f - f_m|^p \, d\mu \right) \\ &\lesssim \int_K |f|^p \, dm + \|f\|_{\mathcal{F}_p}^p \quad \text{(by (9.32) and Jensen's inequality).}\end{aligned} \tag{9.33}$$

Therefore the first estimate in (9.25) follows from (9.23) and (9.24). The proof of the second estimate in (9.25) is similar. □

We observe two important consequences of Theorem 9.22. The first one states that Loewner measures must be minimal energy dominant measures for the self-similar energy $(\mathcal{E}_p, \mathcal{F}_p)$.

Theorem 9.25. *Suppose Assumption 9.16. Then μ is a minimal energy dominant measure for $(\mathcal{E}_p, \mathcal{F}_p)$. Furthermore, there exist $C \in (0,\infty)$ and $u \in \mathscr{C}_p$, we have*

$$C^{-1}\Gamma_p\langle u\rangle(B) \le \mu(B) \le C\Gamma_p\langle u\rangle(B) \quad \text{for all Borel subset } B \subset K. \tag{9.34}$$

Proof. By Theorem 9.22, $\Gamma_p\langle f\rangle \ll \mu$ for all $f \in \mathscr{C}_p$. Combining with the density of $\mathcal{C}(K) \cap \mathcal{F}_p(K,d,m)$ (Theorem 6.17(v)) and Lemma 9.20(a), we obtain the domination property: $\Gamma_p\langle f'\rangle \ll \mu$ for all $f' \in \mathcal{F}_p(K,d,m)$.

By [HKST, Corollary 8.3.16] and a biLipschitz change of metric if necessary, we can assume that θ is a geodesic metric. Consider the function $u(\cdot) = \theta(x_0,\cdot)$ for some $x_0 \in K$. Since u is Lipschitz in (K,θ) and K is compact, we have $u \in N^{1,p}(K,\theta,\mu)$ by [HKST, Lemma 6.2.6]. Furthermore, by considering geodesics in (K,θ), we can show that $\operatorname{lip} u \equiv 1$. By [HKST, Theorem 13.5.1], we have that the minimal p-weak upper gradient g_u of u satisfies $g_u = 1$ μ-almost everywhere. By (9.23) in Theorem 9.22, we have that $\mu \ll \Gamma_p\langle u\rangle$ and hence μ is a minimal energy dominant measure and satisfies (9.34). □

The second one is the identification of the two different Sobolev spaces $\mathcal{F}_p(K,d,m)$ and $N^{1,p}(K,\theta,\mu)$.

Theorem 9.26. *Suppose Assumption 9.16. Then there exists a bounded, linear bijection $\iota\colon \mathcal{F}_p(K,d,m) \to N^{1,p}(K,\theta,\mu)$ satisfying*

$$C_1^{-1}\|f\|_{\mathcal{F}_p} \le \|\iota(f)\|_{N^{1,p}(\mu)} \le C_1\|f\|_{\mathcal{F}_p} \quad \text{for all } f \in \mathcal{F}_p(K,d,m), \tag{9.35}$$

where $C_1 \ge 1$ is the constant in (9.25). Furthermore if $f \in \mathcal{C}(K) \cap \mathcal{F}_p(K,d,m)$, then ι maps the equivalence class containing f in $\mathcal{F}_p(K,d,m)$ to the equivalence class containing f in $N^{1,p}(K,\theta,\mu)$.

Proof. We first note that $\mathscr{C}_p$ is a dense linear subspace of both $\mathcal{F}_p(K,d,m)$ and $N^{1,p}(K,\theta,\mu)$ by Theorem 6.17 and [HKST, Theorem 8.2.1]. Let $\iota_0\colon (\mathscr{C}_p, \|\cdot\|_{\mathcal{F}_p}) \to N^{1,p}(K,\theta,\mu)$ be the inclusion map, i.e., $\iota_0(f) = [f]_{N^{1,p}}$ for $f \in \mathscr{C}_p$, where $[f]_{N^{1,p}}$ is the equivalence class defined in Definition 9.5. By (9.25) in Theorem 9.22, we have $C_1^{-1}\|f\|_{\mathcal{F}_p} \le \|\iota_0(f)\|_{N^{1,p}} \le C_1\|f\|_{\mathcal{F}_p}$ for all $f \in \mathscr{C}_p$. Hence, by [Meg, Proposition 1.4.14], ι_0 is an isomorphism. By [Meg, Theorem 1.9.1] and the density of $\mathscr{C}_p$, there is a unique extension $\iota\colon \mathcal{F}_p(K,d,m) \to N^{1,p}(K,\theta,\mu)$ of ι_0, which is also an isomorphism satisfying $C_1^{-1}\|f\|_{\mathcal{F}_p} \le \|\iota(f)\|_{N^{1,p}} \le C_1\|f\|_{\mathcal{F}_p}$ for all $f \in \mathcal{F}_p(K,d,m)$. □

We conclude this subsection by extending the comparability result of energy measures to all functions in Sobolev spaces through the above isomorphism.

Corollary 9.27. *Suppose Assumption 9.16 and let $\iota\colon \mathcal{F}_p(K,d,m) \to N^{1,p}(K,\theta,\mu)$ be the identification map in Theorem 9.26. Then there exists $C \ge 1$ such that the following hold: For any $f \in \mathcal{F}_p(K,d,m)$ and any Borel set $B \in \mathcal{B}(K)$,*

$$C^{-1}\Gamma_p\langle f\rangle(B) \le \int_B g_{\iota(f)}^p \, d\mu \le C\Gamma_p\langle f\rangle(B). \tag{9.36}$$

In particular,

$$C^{-1}\mathcal{E}_p(f) \le \int_X g_{\iota(f)}^p \, d\mu \le C\mathcal{E}_p(f) \quad \textit{for all } f \in \mathcal{F}_p(K,d,m). \tag{9.37}$$

Proof. By [HKST, (6.3.18)], for any $u, v \in N^{1,p}(K,\theta,\mu)$ and $B \in \mathcal{B}(K)$, we have

$$\left(\int_B g_{u+v}^p \, d\mu\right)^{1/p} \le \left(\int_B g_u^p \, d\mu\right)^{1/p} + \left(\int_B g_v^p \, d\mu\right)^{1/p}.$$

In particular, $\lim_{n\to\infty}\int_B g_{u_n}^p \, d\mu = \int_B g_u^p \, d\mu$ whenever $\lim_{n\to\infty}\|u - u_n\|_{N^{1,p}} = 0$. Let $f \in \mathcal{F}_p(K,d,m)$ and pick a sequence $\{f_n\}_n \subseteq \mathscr{C}_p$ such that $\lim_{n\to\infty}\|f - f_n\|_{\mathcal{F}_p} = 0$. By (9.35), we then have $\lim_{n\to\infty}\|\iota(f) - \iota(f_n)\|_{N^{1,p}} = 0$. Therefore, letting $n \to \infty$ in (9.23) for f_n yields (9.36). □

We are now ready to prove Theorem 1.7.

Proof of Theorem 1.7. Theorems 9.22, 9.26 and Corollary 9.27. The second assertion follows from Theorem 9.25. □

10 Conjectures and open problems

We conclude this paper by mentioning some related open problems and conjectures.

To construct a Hölder continuous cutoff function with low energy and to obtain Poincaré inequality, the condition $\zeta = d_{\mathrm{f}} - d_{\mathrm{w}}(p) < 1$ was crucial. This is because the conclusion of Theorem 3.2 fails without the condition $\zeta < 1$. However, it is conceivable that capacity bounds imply Poincaré inequality without this restriction but such a result would require a very different approach.

Problem 10.1. Relax the conditions $\zeta < 1$ in Theorems 4.2 and 5.6.

Problem 10.1 is similar in spirit to the *resistance conjecture* for the case $p = 2$ and hence it appears very challenging [Mur24, §6.3].

In this paper, we confine ourselves to the planar standard Sierpiński carpet but it is of interest to define Sobolev spaces on other fractals.

Problem 10.2. Construct Sobolev spaces, p-energies, energy measures for other examples such as Sierpiński cross [Kig09], subsystems of (hyper)cubic tiling [Kig23], unconstrained Sierpiński carpets [CQ24, CQ23+], boundaries of hyperbolic groups and Julia sets of conformal dynamical systems [Bon06, Kle06]. Added in revision: Recently, Anttila and Eriksson-Bique [AEB24+b] introduced a new framework of fractal spaces arising from *iterated graph systems* and established the combinatorial Loewner property for these fractals. It would be interesting to construct/investigate Sobolev spaces, p-energies, energy measures in this framework.

In the case of $p = 2$, the Dirichlet form $(\mathcal{E}_2, \mathcal{F}_2)$ constructed in Theorem 1.1 is unique (up to multiplications of constants) by [BBKT, Theorem 1.2]. It is natural to expect that such the uniqueness is true for any $p \in (1, \infty)$.

Conjecture 10.3. For any $p \in (1, \infty)$, self-similar p-energy satisfying the conditions in Theorem 1.1 is unique up to multiplications of constants. We expect that the uniqueness is true for a wide class of Sierpiński carpets (e.g. generalized Sierpiński carpets).

Compared to our $(1, p)$-Sobolev space $\mathcal{F}_p$, the definition of energy measures on a self-similar set heavily depends on the self-similarity. This is a difference from the case $p = 2$ (Dirichlet form theory) and is an obstacle to develop general theory. This motivates the following question.

Problem 10.4. Let (K, d) be a compact metric space satisfying Assumption 6.15. Define p-energy measures $\Gamma_p\langle \cdot \rangle$ on K (without using the self-similarity) and establish their basic properties (e.g. Theorem 1.2(ii),(iii) and (vi)).

Added in revision: There has been recent progress on Problem 10.4. In [KS24+a], by using the Riesz–Markov–Kakutani representation theorem, p-energy measures are constructed for specific p-energy forms called the *Korevaar–Schoen p-energy forms*. See also [ARB24+, Section 4] for a related result on Cheeger spaces.

It is also natural to expect that p-energy measures on typical fractals are mutually singular with the underlying self-similar measures (cf. [Hin05, KM20] for the case $p = 2$).

Problem 10.5. For a self-similar set (K, d) satisfying Assumption 6.15 with $\beta > p$, show that $\Gamma_p\langle f\rangle \perp m$ for any $f \in \mathcal{F}_p$, where m is the self-similar measure.

The next two problems are motivated by a desire to understand the dependence of the Sobolev space $\mathcal{F}_p$ and energy measures on the exponent p.

Problem 10.6. Let $p, q \in (1, \infty)$ be distinct. Let ν_p, ν_q be minimal energy-dominant measures of $(\mathcal{E}_p, \mathcal{F}_p), (\mathcal{E}_q, \mathcal{F}_q)$ respectively. Are ν_p and ν_q mutually singular or absolutely continuous?

We also do not know if there are inclusion relations among $\{\mathcal{F}_p\}_{p>1}$.

Problem 10.7. Let $p, q \in (1, \infty)$ be distinct. Determine the intersection $\mathcal{F}_p \cap \mathcal{F}_q$. In particular, does $\mathcal{F}_p \cap \mathcal{F}_q$ contain any non-constant function?

Towards the attainment problem of the Ahlfors regular conformal dimension, we expect that the following variant of Theorem 1.7(ii) to be useful. This conjecture is an analogue of [KM23, Theorem 6.54].

Conjecture 10.8. Let (K, d, m) be the Sierpiński carpet. Suppose that $d_{\mathrm{ARC}}(K, d)$ is attained. There exists h which is d_{ARC}-harmonic with respect to the self-similar d_{ARC}-energy $\mathcal{E}_{d_{\mathrm{ARC}}}$ on $K \setminus \mathcal{V}_0$ such that $\Gamma_{d_{\mathrm{ARC}}}\langle h\rangle$ is also an optimal measure.

A Some useful results

In this section, we recall Mazur's lemma (Lemma A.1) and prove Lemma 9.23. We start with Mazur's lemma, which is frequently used in the paper.

Lemma A.1 ([HKST, page 19])**.** *Let $(v_i)_{i\in\mathbb{N}}$ be a sequence in a normed space V converging weakly to some element $v \in V$. Then there exist a strictly increasing sequence $\{l_n\}_{n\geq 1}$ of positive integers with $l_n \geq n$, and, for each $n \geq 1$, $(\lambda_{i,n})_{i=n}^{l_n} \in [0, 1]^{l_n-n+1}$ with $\sum_{i=n}^{l_n} \lambda_{i,n} = 1$ such that $\sum_{i=n}^{l_n} \lambda_{i,n} v_i$ converges strongly to v as $n \to \infty$.*

To prove Lemma 9.23, we will use the following version of a Whitney cover.

Definition A.2 ([Mur24, Definition 2.3])**.** Let (X, d) be a metric space and $\varepsilon \in (0, 1/2)$. Let U be a non-empty proper subset of X such that $U \neq X$. A collection of balls $\mathfrak{R} = \{B(x_i, r_i) \mid x_i \in U, r_i > 0, i \in I\}$ is said to be an ε-*Whitney cover* of U if it satisfies the following conditions:

(1) The balls in $\mathfrak{R}$ are pairwise disjoint.

(2) The radius r_i satisfies

$$r_i = \frac{\varepsilon}{1+\varepsilon}\,\mathrm{dist}(x_i, X \setminus U), \quad \text{for each } i \in I. \tag{A.1}$$

(3) It holds that $\bigcup_{i \in I} B(x_i, 2(1+\varepsilon)r_i) = U$.

Remark A.3. From (A.1), we observe that $B\bigl(x_i, \varepsilon^{-1}(1+\varepsilon)r_i\bigr) \subseteq U$ for all $i \in I$.

The existence of such an ε-Whitney cover of any non-empty open subset U of a given metric space (X, d) for all $\varepsilon \in (0, 1/2)$ is ensured by [Mur24, Proposition 3.2(a)]. The following proposition states a basic overlapping property of Whitney covers on a doubling metric space.

Proposition A.4 ([Mur24, Proposition 3.2(d)]). *Let (X, d) be a metric space and let U be a non-empty proper subset of X such that $U \neq X$. If (X, d) is metric doubling, then for any $\varepsilon \in (0, 1/2)$ there exists $C > 0$ (depending only on ε and the doubling constant of (X, d)) such that the following hold: for any ε-Whitney cover $\mathfrak{R} = \{B(x_i, r_i) \mid x_i \in U, r_i > 0, i \in I\}$ of U, we have $\sum_{i \in I} \mathbb{1}_{B(x_i, \varepsilon^{-1} r_i)} \leq C$.*

Now we can prove the desired lemma:

Proof of Lemma 9.23. By the outer regularity of measures ν_1 and ν_2 [HKST, Proposition 3.3.37], it suffices to verify (9.26) for all open sets.

To this end, let U be an arbitrary non-empty open subset of X. Let us fix small enough ε so that $0 < \varepsilon < (3A_1)^{-1}$ and choose a ε-Whitney cover $\mathfrak{R} = \{B(x_i, r_i) \mid x_i \in U, r_i > 0, i \in I\}$ of U. Then we note that $B(x_i, 3A_1 r_i) \subseteq U$ for all $i \in I$. By the bounded overlap property Proposition A.4, there exists C_2 depending only on C_1, A_1 and the constant associated to the doubling property of (X, d) such that

$$\nu_1(U) \leq \sum_{B(x_i, r_i) \in \mathfrak{R}} \nu_1(B(x_i, 3r_i)) \leq \sum_{B(x_i, r_i) \in \mathfrak{R}} C_1 \nu_2(B(x_i, 3A_1 r_i)) \leq C_2 \nu_2(U), \tag{A.2}$$

which concludes the proof. □

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Mathav Murugan
Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada.
E-mail: mathav@math.ubc.ca

Ryosuke Shimizu (JSPS Research Fellow PD)
Waseda Research Institute for Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan.
E-mail: r-shimizu@aoni.waseda.jp