Contraction properties and differentiability of p-energy forms with applications to nonlinear potential theory on self-similar sets

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Abstract

We introduce new contraction properties called the *generalized p-contraction property* for p-energy forms as generalizations of many well-known inequalities, such as Clarkson's inequalities, the strong subadditivity and the "Markov property" in the theory of nonlinear Dirichlet forms, and show that any p-energy form satisfying Clarkson's inequalities is Fréchet differentiable. We also verify the generalized p-contraction property for p-energy forms constructed by Kigami [*Mem. Eur. Math. Soc.* **5** (2023)] and by Cao–Gu–Qiu [*Adv. Math.* **405** (2022), no. 108517]. As a general framework of p-energy forms taking into consideration the generalized p-contraction property, we introduce the notion of *p-resistance form* and investigate fundamental properties for p-harmonic functions with respect to p-resistance forms. In particular, some new estimates on scaling factors of p-energy forms are obtained by establishing Hölder regularity estimates for harmonic functions, and the p-walk dimensions of the generalized Sierpiński carpets and D-dimensional level-l Sierpiński gasket are shown to be strictly greater than p.

Keywords: generalized p-contraction property, p-resistance form, p-harmonic function, p-energy measure, self-similar p-energy form

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Contents

1 Introduction

In the field of 'analysis on fractals', on a large class of self-similar sets including the Sierpiński gasket and the Sierpiński carpet (see Figure 1.1), it is an established result that there exists a nice Dirichlet form $(\mathcal{E}_2, \mathcal{F}_2)$, which is an analogue of the pair of the Dirichlet 2-energy $\int |\nabla u|^2 \, dx$ and the associated $(1, 2)$-Sobolev space $W^{1,2}$ on a differentiable space (see, e.g., [Kig01, BB99]). Once we obtain a nice Dirichlet form, the theory of symmetric Dirichlet forms provides us the associated energy measure $\Gamma_2\langle u \rangle$ playing the role of $|\nabla u|^2 \, dx$ whose existence is highly non-trivial because the density "$|\nabla u|$" usually does not make sense on fractals (see [Hin05, KM20]). The main purpose of this article is to develop a general theory for L^p-analogues of $(\mathcal{E}_2, \mathcal{F}_2, \Gamma_2\langle \cdot \rangle)$, where $p \in (1, \infty)$, on the basis of the new contraction property called the *generalized p-contraction property*. To state results precisely, throughout this introduction, we fix a self-similar set K and a natural Hausdorff measure m on K. For a large class of the pair (K, p), a natural L^p-analogue of $(\mathcal{E}_2, \mathcal{F}_2)$ on K, namely a p-energy form $(\mathcal{E}_p, \mathcal{F}_p)$ playing the role of $\int |\nabla u|^p \, dx$ and the associated $(1, p)$-Sobolev space $W^{1,p}$, where $\mathcal{F}_p$ is a linear subspace of $L^p(K, m)$ and $\mathcal{E}_p \colon \mathcal{F}_p \to [0, \infty)$ is p-homogeneous in the sense that $\mathcal{E}_p(au) = |a|^p \, \mathcal{E}_p(u)$ for any $a \in \mathbb{R}$ and any $u \in \mathcal{F}_p$, have been constructed in several works [HPS04, Kig23, Shi24, CGQ22, MS23+, KO+][1], most of which are very recent. Furthermore, the associated p-energy measure $\Gamma_p\langle u \rangle$, which is a finite Borel measure on K and an analogue of $|\nabla u|^p \, dx$, has been introduced in [Shi24, MS23+] with the help of the self-similarity of $(\mathcal{E}_p, \mathcal{F}_p)$. See Section 5 for details on the self-similarity of a p-energy form, and Example 4.2 for examples of p-energy measures without relying on the self-similarity. Compared with the case $p = 2$, where the theory of symmetric Dirichlet forms is applicable, very little has been established to deal with $(\mathcal{E}_p, \mathcal{F}_p, \Gamma_p\langle \cdot \rangle)$ in a general framework. In particular, there are two missing pieces in known results of $(\mathcal{E}_p, \mathcal{F}_p, \Gamma_p\langle \cdot \rangle)$: first, useful contraction properties of it, and secondly, the (Fréchet) differentiabilities of $\mathcal{E}_p$ and of Γ_p. In the first half of this paper (Sections 2-5), we aim to establish general results filling these missing pieces. We shall explain in more detail below.

The first missing piece is contraction properties for $(\mathcal{E}_p, \mathcal{F}_p, \Gamma_p\langle \cdot \rangle)$. Every p-energy form $(\mathcal{E}_p, \mathcal{F}_p)$ constructed in the previous studies is known to satisfy the following *unit contractivity:*

$$u^+ \wedge 1 \in \mathcal{F}_p \text{ and } \mathcal{E}_p(u^+ \wedge 1) \leq \mathcal{E}_p(u) \text{ for any } u \in \mathcal{F}_p. \tag{1.1}$$

In the case $p = 2$, by using some helpful expressions of $\mathcal{E}_2$, e.g. [FOT, Lemma 1.3.4 and (3.2.12)], (1.1) can be improved to the following *normal contractivity* (see [MR, Chapter I, Theorem 4.1.2] for example): if $n \in \mathbb{N}$, $T \colon \mathbb{R}^n \to \mathbb{R}$ satisfy $|T(x)| \leq \sum_{k=1}^{n} |x_k|$ and $|T(x) - T(y)| \leq \sum_{k=1}^{n} |x_k - y_k|$ for any $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, then for

[1] The differences among these works are the class of (K, p) on which $(\mathcal{E}_p, \mathcal{F}_p)$ is constructed. Let us clarify only some important differences (see [KS23+, Introduction] for details). In [HPS04, CGQ22], K is assumed to be a post-critically finite self-similar set (see Definition 5.3) so that the Sierpiński gasket is included while the Sierpiński carpet is excluded. The case K is the Sierpiński carpet is allowed in [Kig23, Shi24, MS23+, KO+], but we need to assume that p is strictly greater than the Ahlfors regular conformal dimension of K (see Definition 8.5-(4)) in [Kig23, Shi24, KO+].

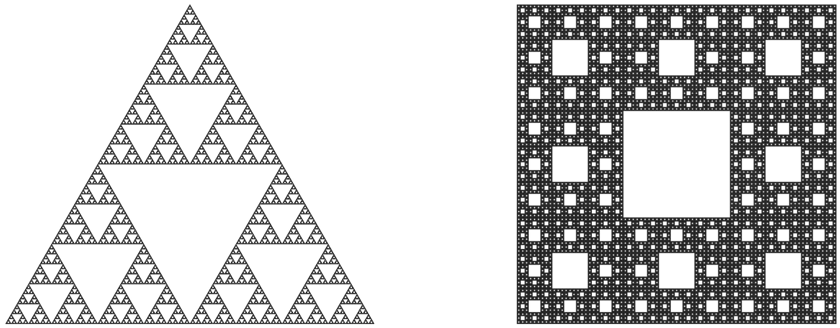

Figure 1.1: The Sierpiński gasket (left) and the Sierpiński carpet (right)

any $\boldsymbol{u} = (u_1, \dots, u_n) \in \mathcal{F}_2^n$ we have

$$T(\boldsymbol{u}) \in \mathcal{F}_2 \quad \text{and} \quad \mathcal{E}_2(T(\boldsymbol{u}))^{\frac{1}{2}} \leq \sum_{k=1}^{n} \mathcal{E}_2(u_k)^{\frac{1}{2}}. \tag{1.2}$$

It is natural to expect that $(\mathcal{E}_p, \mathcal{F}_p)$ also has a similar property to (1.2) since $\mathcal{E}_p(u)$ is an analogue of $\int |\nabla u|^p \, dx$; nevertheless, it is not clear whether (1.1) can be improved in such a way without going back to the constructions of $(\mathcal{E}_p, \mathcal{F}_p)$ in the previous studies. Not only (1.2) but also part of useful inequalities like the following *strong subadditivity* and *p-Clarkson's inequalities*, was not mentioned in [HPS04, Kig23, Shi24, CGQ22, MS23+]:

(Strong subadditivity) For any $u, v \in \mathcal{F}_p$, we have $u \wedge v, u \vee v \in \mathcal{F}_p$ and

$$\mathcal{E}_p(u \wedge v) + \mathcal{E}_p(u \vee v) \leq \mathcal{E}_p(u) + \mathcal{E}_p(v). \tag{1.3}$$

(p-Clarkson's inequality) For any $u, v \in \mathcal{F}_p$,

$$\begin{cases} \mathcal{E}_p(u+v)^{\frac{1}{p-1}} + \mathcal{E}_p(u-v)^{\frac{1}{p-1}} \leq 2\big(\mathcal{E}_p(u) + \mathcal{E}_p(v)\big)^{\frac{1}{p-1}} & \text{if } p \in (1, 2], \\ \mathcal{E}_p(u+v) + \mathcal{E}_p(u-v) \leq 2\big(\mathcal{E}_p(u)^{\frac{1}{p-1}} + \mathcal{E}_p(v)^{\frac{1}{p-1}}\big)^{p-1} & \text{if } p \in (2, \infty). \end{cases} \tag{1.4}$$

These inequalities play significant roles in the *nonlinear potential theory* with respect to $(\mathcal{E}_p, \mathcal{F}_p)$. For example, (1.3) will be important to consider the p-capacity associated with $(\mathcal{E}_p, \mathcal{F}_p)$; see [BV05, (H3)]. Also, we frequently use (1.4) in this paper; see Theorem 1.3 below for one of the most important consequence of (1.4). Since we do not know whether the property (1.1) is enough for desirable inequalities unlike the case $p = 2$, one needs to go back the constructions of $(\mathcal{E}_p, \mathcal{F}_p)$ in the preceding works if one wishes to show them. The situation is similar for p-energy measures. It is natural to expect that p-energy measures inherit contraction properties from $(\mathcal{E}_p, \mathcal{F}_p)$, however, in order to show such a property for p-energy measures, we need to recall how p-energy measures are constructed partially because no canonical way to define p-energy measures is known (see [MS23+, Problem 12.5]).

To overcome this situation, in this paper, we will introduce the following notion of *generalized p-contraction property* as a candidate of the strongest possible form of contraction properties of p-energy forms.

Definition 1.1 (Generalized p-contraction property; see also Definition 2.1)**.** Let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,p]$ and $q_2 \in [p,\infty]$. We say that $(\mathcal{E}_p, \mathcal{F}_p)$ satisfies the *generalized p-contraction property* if $T = (T_1, \dots, T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfies $T(0) = 0$ and $\|T(x) - T(y)\|_{\ell^{q_2}} \leq \|x - y\|_{\ell^{q_1}}$ for any $x, y \in \mathbb{R}^{n_1}$, then for any $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}_p^{n_1}$ we have

$$T(\boldsymbol{u}) \in \mathcal{F}_p^{n_2} \quad \text{and} \quad \left\| \left(\mathcal{E}_p(T_l(\boldsymbol{u}))^{\frac{1}{p}} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \leq \left\| \left(\mathcal{E}_p(u_k)^{\frac{1}{p}} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \tag{1.5}$$

Note that the case $(p, n_1, n_2, q_1, q_2) = (2, n, 1, 1, p)$ is the same as (1.2) for symmetric Dirichlet forms. As recorded in the following proposition, (1.5) is actually a generalization of many useful inequalities like (1.2), (1.3) and (1.4).

Proposition 1.2 (Proposition 2.2)**.** *Let $\varphi \in C(\mathbb{R})$ satisfy $\varphi(0) = 0$ and $|\varphi(t) - \varphi(s)| \leq |t - s|$ for any $s, t \in \mathbb{R}$. Assume that $(\mathcal{E}_p, \mathcal{F}_p)$ satisfies the generalized p-contraction property. Then the following hold.*

(a) (Triangle inequality and strict convexity) *$\mathcal{E}_p^{1/p}$ is a seminorm on $\mathcal{F}_p$, and for any $\lambda \in (0,1)$ and any $f, g \in \mathcal{F}_p$ with $\mathcal{E}_p(f) \wedge \mathcal{E}_p(g) \wedge \mathcal{E}_p(f - g) > 0$,*

$$\mathcal{E}_p(\lambda f + (1-\lambda) g) < \lambda \mathcal{E}_p(f) + (1-\lambda)\mathcal{E}_p(g).$$

(b) (Lipschitz contractivity) *$\varphi(u) \in \mathcal{F}_p$ and $\mathcal{E}_p(\varphi(u)) \leq \mathcal{E}_p(u)$ for any $u \in \mathcal{F}_p$.*

(c) (Strong subadditivity) *Assume that φ is non-decreasing. Then for any $f, g \in \mathcal{F}_p$,*

$$\mathcal{E}_p\bigl(f - \varphi(f - g)\bigr) + \mathcal{E}_p\bigl(g + \varphi(f - g)\bigr) \leq \mathcal{E}_p(f) + \mathcal{E}_p(g).$$

In particular, (1.3) holds.

(d) (Leibniz rule) *For any $f, g \in \mathcal{F}_p \cap L^\infty(K, m)$, we have*

$$f \cdot g \in \mathcal{F}_p \quad \text{and} \quad \mathcal{E}_p(f \cdot g)^{\frac{1}{p}} \leq \|g\|_{L^\infty(K,m)} \mathcal{E}_p(f)^{\frac{1}{p}} + \|f\|_{L^\infty(K,m)} \mathcal{E}_p(g)^{\frac{1}{p}}.$$

(e) (p-Clarkson's inequality) *Let $f, g \in \mathcal{F}_p$. If $p \in (1,2]$, then*

$$2\bigl(\mathcal{E}_p(f)^{\frac{1}{p-1}} + \mathcal{E}_p(g)^{\frac{1}{p-1}}\bigr)^{p-1} \leq \mathcal{E}_p(f+g) + \mathcal{E}_p(f-g) \leq 2\bigl(\mathcal{E}_p(f) + \mathcal{E}_p(g)\bigr).$$

If $p \in [2,\infty)$, then

$$2\bigl(\mathcal{E}_p(f) + \mathcal{E}_p(g)\bigr) \leq \mathcal{E}_p(f+g) + \mathcal{E}_p(f-g) \leq 2\bigl(\mathcal{E}_p(f)^{\frac{1}{p-1}} + \mathcal{E}_p(g)^{\frac{1}{p-1}}\bigr)^{p-1}.$$

In particular, (1.4) holds.

Since the generalized p-contraction property is introduced as arguably the strongest possible formulation of the contraction property of $(\mathcal{E}_p, \mathcal{F}_p)$, it is highly non-trivial whether p-energy forms constructed in the previous studies satisfy it. In Section 8, we will see that we can still verify the existing constructions of p-energy forms in the previous studies so as to get ones satisfying (1.5). (See also [KS.a] for an approach, which is based on Korevaar–Schoen p-energy forms, to obtain p-energy forms satisfying (1.5).)

The other missing piece is the differentiability of p-energy forms, which should be useful to study *p-harmonic functions* with respect to $\mathcal{E}_p$. (See [KM23, Problem 7.7] and [MS23+, Conjecture 10.9] for some motivations to investigate p-harmonic functions on fractals.) In [HPS04, Shi24, CGQ22], p-harmonic functions are defined as functions minimizing $\mathcal{E}_p$ under some fixed boundary conditions. However, it is still unclear how to give an equivalent definition of p-harmonic function in a weak sense due to the lack of 'two-variable version' $\mathcal{E}_p(u;\varphi)$ [Kig23, Problem 2 in Section 6.3]. We shall recall the Euclidean case to explain the importance of this object. Let $D \in \mathbb{N}$ and $U \subseteq \mathbb{R}^D$ a domain. A function $u \in W^{1,p}(\mathbb{R}^D)$ is said to be p-harmonic on U in the weak sense if

$$\int_{\mathbb{R}^D} |\nabla u(x)|^{p-2} \langle \nabla u(x), \nabla\varphi(x)\rangle_{\mathbb{R}^D}\, dx = 0 \quad \text{for every } \varphi \in C_c^\infty(U), \tag{1.6}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{R}^D}$ is the inner product of $\mathbb{R}^D$. It is well known that (1.6) is equivalent to

$$\int_{\mathbb{R}^D} |\nabla u(x)|^p\, dx = \inf\left\{ \int_{\mathbb{R}^D} |\nabla v(x)|^p\, dx \,\middle|\, v \in W^{1,p}(\mathbb{R}^D), u - v \in W_0^{1,p}(U) \right\}. \tag{1.7}$$

An issue to consider an analogue of (1.6) in terms of $\mathcal{E}_p$ is that we do not have a satisfactory counterpart, $\mathcal{E}_p(u;\varphi)$, of $\int |\nabla u|^{p-2} \langle \nabla u, \nabla\varphi\rangle\, dx$ associated with $\mathcal{E}_p$. As mentioned in [SW04, (2.1)], the ideal definition of $\mathcal{E}_p(u;\varphi)$[2] is

$$\mathcal{E}_p(u;\varphi) \coloneqq \frac{1}{p}\frac{d}{dt}\mathcal{E}_p(u+t\varphi)\Big|_{t=0}, \tag{1.8}$$

but the existence of this derivative is unclear[3] because the constructions of $\mathcal{E}_p$ in the previous studies include many steps such as the operation of taking a subsequential scaling limit of discrete p-energies. Similarly, in respect of p-energy measures, no suitable way is known to define a 'two-variable version' $\Gamma_p\langle u;\varphi\rangle$, which plays the role of $|\nabla u|^{p-2}\langle \nabla u, \nabla\varphi\rangle\, dx$. The ideal definition of $\Gamma_p\langle u;\varphi\rangle$ is similar to (1.8), i.e., for any Borel set A of K,

$$\Gamma_p\langle u;\varphi\rangle(A) \coloneqq \frac{1}{p}\frac{d}{dt}\Gamma_p\langle u+t\varphi\rangle(A)\Big|_{t=0}. \tag{1.9}$$

Such a signed measure is discussed in [BV05, Section 5], but the existence of the derivative in (1.9) is assumed in [BV05] (in some uniform manner); see [BV05, (H4) and the beginning of Section 5] for details. In [Cap07], the (scale-invariant) elliptic Harnack inequality for p-harmonic functions on *metric fractals* (see [Cap07, Definition 2.3]) is proved under some assumptions including the existence of $\Gamma_p\langle u;\varphi\rangle$, which is called the *measure-valued p-Lagrangian* and denoted by $\mathcal{L}^{(p)}(u,\varphi)$ in [Cap07], as in [BV05]. However, in the case that there is no explicit expression of the p-energy measure $\Gamma_p\langle u\rangle$ unlike the case

[2] Strichartz and Wong [SW04] have proposed an approach based on *subderivative* instead of (1.8), i.e., $\mathcal{E}_p(u;\varphi)$ is defined as the interval $\left[\mathcal{E}_p^-(u;\varphi), \mathcal{E}_p^+(u;\varphi)\right]$ where $\frac{d^\pm}{dt}\mathcal{E}_p(u+t\varphi)\Big|_{t=0} =: \mathcal{E}_p^\pm(u;\varphi)$.

[3] The case $p = 2$ is special because of the parallelogram law. Indeed, $\mathcal{E}_2$ is known to be a quadratic form and hence $\mathcal{E}_2(u,v) \coloneqq 4^{-1}(\mathcal{E}_2(u+v) - \mathcal{E}_2(u-v))$ is a symmetric form satisfying (1.8).

of Euclidean spaces, there is no proof of the existence of the derivatives in (1.9) in the literature. (The p-energy form on the Sierpiński gasket constructed in [HPS04] is discussed in [Cap07, Section 5] as a concrete examples and it is stated that "we can define the corresponding Lagrangian $\mathcal{L}^{(p)}(u,v)$" in p. 1315 of that paper, nevertheless, we have been unable to find in the literature a rigorous proof of the existence of the derivatives in [Cap07, p. 1315] defining $\mathcal{E}_g(u,v)$ and in [Cap07, p. 1303, (L5)] defining $\mathcal{L}^{(p)}(u,v)$ for the p-energy form on the Sierpinski gasket obtained in [HPS04].)

As another main contributions of this paper, we make a key observation that p-Clarkson's inequality (1.4) implies the desired differentiability of $\mathcal{E}_p$. In addition to this result, we record basic properties of $\mathcal{E}_p(u;\varphi)$ given by (1.8) in the following theorem.

Theorem 1.3 (Proposition 3.5 and Theorem 3.6)**.** *Assume that $(\mathcal{E}_p,\mathcal{F}_p)$ satisfies (1.4). Then $\mathbb{R}\ni t\mapsto \mathcal{E}_p(f+tg)\in[0,\infty)$ is differentiable for any $f,g\in\mathcal{F}_p$, and for any $s\in\mathbb{R}$,*

$$\lim_{\delta\downarrow 0}\sup_{g\in\mathcal{F}_p;\mathcal{E}_p(g)\leq 1}\left|\frac{\mathcal{E}_p(f+(s+\delta)g)-\mathcal{E}_p(f+sg)}{\delta}-\frac{d}{dt}\mathcal{E}_p(f+tg)\bigg|_{t=s}\right|=0.$$

We define $\mathcal{E}_p(\,\cdot\,;\,\cdot\,)\colon \mathcal{F}_p\times\mathcal{F}_p\to\mathbb{R}$ by $\mathcal{E}_p(f;g):=\frac{1}{p}\frac{d}{dt}\mathcal{E}_p(f+tg)\big|_{t=0}$. Let $a\in\mathbb{R}$, $f,f_1,f_2,g\in\mathcal{F}_p$ and $h\in\mathcal{E}_p^{-1}(0)$. Then the following hold.

(a) *$\mathcal{E}_p(f;f)=\mathcal{E}_p(f)$ and $\mathcal{E}_p(af;g)=\operatorname{sgn}(a)\,|a|^{p-1}\,\mathcal{E}_p(f;g)$.*
(b) *The map $\mathcal{E}_p(f;\,\cdot\,)\colon\mathcal{F}_p\to\mathbb{R}$ is linear.*
(c) *$\mathcal{E}_p(f;h)=0$ and $\mathcal{E}_p(f+h;g)=\mathcal{E}_p(f;g)$.*
(d) *$\mathbb{R}\ni t\mapsto\mathcal{E}_p(f+tg;g)\in\mathbb{R}$ is strictly increasing if and only if $g\notin\mathcal{E}_p^{-1}(0)$*
(e) *$|\mathcal{E}_p(f;g)|\leq\mathcal{E}_p(f)^{\frac{p-1}{p}}\mathcal{E}_p(g)^{\frac{1}{p}}$.*
(f) *$|\mathcal{E}_p(f_1;g)-\mathcal{E}_p(f_2;g)|\leq C_p\big(\mathcal{E}_p(f_1)\vee\mathcal{E}_p(f_2)\big)^{\frac{p-1-\alpha_p}{p}}\mathcal{E}_p(f_1-f_2)^{\frac{\alpha_p}{p}}\mathcal{E}_p(g)^{\frac{1}{p}}$, where $\alpha_p=\frac{1}{p}\wedge\frac{p-1}{p}$ and some constant $C_p\in(0,\infty)$ determined solely and explicitly by p.*

We also establish a similar result for p-energy measures as follows, which is the first rigorous result on the existence of the derivative in (1.9) for p-energy measures on fractals.

Theorem 1.4 (Propositions 4.3, 4.8 and Theorem 4.5)**.** *Assume that $\{\Gamma_p\langle u\rangle\}_{u\in\mathcal{F}_p}$ satisfies*

$$\begin{cases}\Gamma_p\langle f+g\rangle(A)^{\frac{1}{p-1}}+\Gamma_p\langle f-g\rangle(A)^{\frac{1}{p-1}}\leq 2\big(\Gamma_p\langle f\rangle(A)+\Gamma_p\langle g\rangle(A)\big)^{\frac{1}{p-1}} & \text{if } p\in(1,2],\\ \Gamma_p\langle f+g\rangle(A)+\Gamma_p\langle f-g\rangle(A)\leq 2\big(\Gamma_p\langle f\rangle(A)^{\frac{1}{p-1}}+\Gamma_p\langle g\rangle(A)^{\frac{1}{p-1}}\big)^{p-1} & \text{if } p\in(2,\infty),\end{cases}$$

for any Borel set A of K. Then $\mathbb{R}\ni t\mapsto\Gamma_p\langle f+tg\rangle(A)\in[0,\infty)$ is differentiable for any $f,g\in\mathcal{F}_p$ and any Borel set A of K, and for any $s\in\mathbb{R}$,

$$\lim_{\delta\downarrow 0}\sup_{g\in\mathcal{F}_p;\mathcal{E}_p(g)\leq 1}\left|\frac{\Gamma_p\langle f+(s+\delta)g\rangle(A)-\Gamma_p\langle f+sg\rangle(A)}{\delta}-\frac{d}{dt}\Gamma_p\langle f+tg\rangle(A)\bigg|_{t=s}\right|=0.$$

We define $\Gamma_p\langle f;g\rangle(A):=\frac{1}{p}\frac{d}{dt}\Gamma_p\langle f+tg\rangle(A)\big|_{t=0}$. Then $\Gamma_p\langle f;g\rangle$ is a signed Borel measure on K. Moreover, for any Borel set A of K, $\Gamma_p\langle\,\cdot\,;\,\cdot\,\rangle(A)\colon\mathcal{F}_p\times\mathcal{F}_p\to\mathbb{R}$ satisfies the following properties: Let $a\in\mathbb{R}$, $f,f_1,f_2,g,h\in\mathcal{F}_p$ with $\Gamma_p\langle h\rangle(A)=0$. Then

(a) *$\Gamma_p\langle f; f\rangle(A) = \Gamma_p\langle f\rangle$ and $\Gamma_p\langle af; g\rangle(A) = \mathrm{sgn}(a)\,|a|^{p-1}\,\Gamma_p\langle f; g\rangle(A)$.*
(b) *The map $\Gamma_p\langle f; \cdot\,\rangle(A) \colon \mathcal{F}_p \to \mathbb{R}$ is linear.*
(c) *$\Gamma_p\langle f; h\rangle(A) = 0$ and $\Gamma_p\langle f + h; g\rangle(A) = \Gamma_p\langle f; g\rangle(A)$.*
(d) *$\mathbb{R} \ni t \mapsto \Gamma_p\langle f + tg; g\rangle(A) \in \mathbb{R}$ is strictly increasing if and only if $\Gamma_p\langle g\rangle(A) > 0$*
(e) *For any Borel measurable functions $\varphi, \psi \colon K \to [0, \infty]$,*

$$\int_K \varphi\psi\, d\,|\Gamma_p\langle f; g\rangle| \le \left(\int_K \varphi^{\frac{p}{p-1}}\, d\Gamma_p\langle f\rangle\right)^{(p-1)/p} \left(\int_K \psi^p\, d\Gamma_p\langle g\rangle\right)^{1/p}.$$

(f) *Let $\alpha_p = \frac{1}{p} \wedge \frac{p-1}{p}$. There exists a constant $C_p \in (0, \infty)$ determined solely and explicitly by p such that*

$$\begin{aligned}&|\Gamma_p\langle f_1; g\rangle(A) - \Gamma_p\langle f_2; g\rangle(A)| \\ &\le C_p\bigl(\Gamma_p\langle f_1\rangle(A) \vee \Gamma_p\langle f_2\rangle(A)\bigr)^{\frac{p-1-\alpha_p}{p}}\,\Gamma_p\langle f_1 - f_2\rangle(A)^{\frac{\alpha_p}{p}}\,\Gamma_p\langle g\rangle(A)^{\frac{1}{p}}.\end{aligned}$$

In the second part of this paper (Sections 6 and 7), we aim to develop a general theory for $(\mathcal{E}_p, \mathcal{F}_p)$ on the basis of the generalized p-contraction property and Theorem 1.3. As a satisfactory theory of p-energy forms taking into the generalized p-contraction property and focusing on a "low-dimensional" setting, we will introduce the notion of *p-resistance form*, which can be regarded as a natural L^p-analogue of the theory of resistance forms developed by Kigami mainly in [Kig01, Kig12].

Definition 1.5 (p-Resistance form; see Definition 6.1)**.** $(\mathcal{E}_p, \mathcal{F}_p)$ is said to be a *p-resistance form* on K if and only if it satisfies the following conditions:

(RF1)$_p$ $\mathcal{F}_p$ is a linear subspace of $\mathbb{R}^K$ containing $\mathbb{R}\mathbb{1}_K$ and $\mathcal{E}_p(\,\cdot\,)^{1/p}$ is a seminorm on $\mathcal{F}_p$ satisfying $\{u \in \mathcal{F}_p \mid \mathcal{E}_p(u) = 0\} = \mathbb{R}\mathbb{1}_K$.
(RF2)$_p$ The quotient normed space $(\mathcal{F}_p/\mathbb{R}\mathbb{1}_K, \mathcal{E}_p(\,\cdot\,)^{1/p})$ is a Banach space.
(RF3)$_p$ If $x \ne y \in K$, then there exists $u \in \mathcal{F}_p$ such that $u(x) \ne u(y)$.
(RF4)$_p$ For any $x, y \in K$,

$$R_{\mathcal{E}_p}(x, y) := \sup\left\{ \frac{|u(x) - u(y)|^p}{\mathcal{E}_p(u)} \;\middle|\; u \in \mathcal{F}_p \setminus \mathbb{R}\mathbb{1}_K \right\} < \infty.$$

(RF5)$_p$ $(\mathcal{E}_p, \mathcal{F}_p)$ satisfies the generalized p-contraction property.

We will verify that p-energy forms constructed by Kigami in [Kig23, Theorem 3.21] under the assumptions that the underlying compact metric space is *p-conductively homogeneous* (Definition 8.11) and p is strictly greater than the Ahlfors regular conformal dimension of the underlying space, are p-resistance forms. In addition, we prove that p-energy forms on post-critically finite self-similar sets constructed by Cao–Gu–Qiu in [CGQ22, Proposition 5.3] turn out to be p-resistance forms for any $p \in (1, \infty)$ under the condition (**R**) in [CGQ22, p. 18]. (See Section 8 for details.) Similar to the case $p = 2$, developing a general theory for p-resistance forms allows us to to investigate p-energy forms provided by these broad frameworks in a synthetic manner.

It is immediate that if $(\mathcal{E}_p, \mathcal{F}_p)$ is a p-resistance form on K, then $R_{\mathcal{E}_p}(\,\cdot\,,\,\cdot\,)^{1/p}$ is a metric on K and any function in $\mathcal{F}_p$ is a continuous function on K with respect to the topology induced by this metric. In the theory of resistance forms ($p = 2$), it is well known that $R_{\mathcal{E}_2}(\,\cdot\,,\,\cdot\,)$ is a metric, which is called the *resistance metric* associated with the resistance form $(\mathcal{E}_2, \mathcal{F}_2)$. See [Kig01, Theorem 2.3.4] for a proof. In view of this fact in the case $p = 2$, it is natural to seek the optimal exponent q where $R_{\mathcal{E}_p}(\,\cdot\,,\,\cdot\,)^q$ is a metric. The following theorem gives the answer.

Theorem 1.6 (Corollary 6.32)**.** *If $(\mathcal{E}_p, \mathcal{F}_p)$ is a p-resistance form on K, then $R_{\mathcal{E}_p}(\,\cdot\,,\,\cdot\,)^{\frac{1}{p-1}}$ is a metric on K.*

The power $1/(p-1)$ in the theorem above is sharp; see Example 6.34. Let us call $R_{\mathcal{E}_p}(\,\cdot\,,\,\cdot\,)^{\frac{1}{p-1}}$ the *p-resistance metric* associated with $(\mathcal{E}_p, \mathcal{F}_p)$. The proof of the triangle inequality for the p-resistance metric is done independently by [Her10, ACFP19] for finite weighted graphs (see also [Shi21] for infinite graphs). Our result (Theorem 1.6) is the first result including continuous settings.

We also investigate p-harmonic functions with respect to p-resistance forms, which should correspond to a part of *nonlinear potential theory* where each point has a positive p-capacity. Let us explain some basic results in this introduction. The following definition is a natural analogue of (1.6) (or of (1.7)).

Definition 1.7 ($\mathcal{E}_p$-Harmonic function; see Definition 6.12)**.** Let $(\mathcal{E}_p, \mathcal{F}_p)$ be a p-resistance form on K and let B be a non-empty subset of K. A function $h \in \mathcal{F}_p$ is said to be $\mathcal{E}_p$-*harmonic* on $K \setminus B$ if and only if

$$\mathcal{E}_p(h; \varphi) = 0 \text{ for any } \varphi \in \mathcal{F}_p \text{ with } \varphi|_B = 0,$$

or equivalently

$$\mathcal{E}_p(h) = \inf\{\mathcal{E}_p(u) \mid u \in \mathcal{F}_p, u|_B = h|_B\}.$$

(See Proposition 6.11 for this equivalence.)

A standard argument in variational analysis ensures the existence and the uniqueness of $\mathcal{E}_p$-harmonic function satisfying a given boundary condition.

Proposition 1.8 (see Theorem 6.13)**.** *Let $(\mathcal{E}_p, \mathcal{F}_p)$ be a p-resistance form on K and let B be a non-empty subset of K. We define $\mathcal{F}_p|_B \coloneqq \{u|_B \mid u \in \mathcal{F}_p\}$. Then for any $u \in \mathcal{F}_p|_B$, there exists a unique function $h_B^{\mathcal{E}_p}[u]$ in $\mathcal{F}_p$ satisfying $h_B^{\mathcal{E}_p}[u]\big|_B = u$ and $\mathcal{E}_p(h_B^{\mathcal{E}_p}[u]) = \inf\{\mathcal{E}_p(v) \mid v \in \mathcal{F}_p, v|_B = u\}$.*

Using the (nonlinear) operator $h_B^{\mathcal{E}_p}[\,\cdot\,] \colon \mathcal{F}_p|_B \to \mathcal{F}_p$ given in the proposition above, we can introduce a new p-resistance form on the boundary set, which is called the *trace* of the p-resistance form to the boundary set. This notion is at the core of our theory of p-resistance forms, and turns out to be a powerful tool especially when we work on post-critically finite self-similar sets; see Subsection 8.3 for example. Here we just record fundamental results on traces in the following theorem.

Theorem 1.9 (Trace of p-resistance form; see Theorem 6.13)**.** *Let $(\mathcal{E}_p, \mathcal{F}_p)$ be a p-resistance form on K and let B be a non-empty subset of K. We define $\mathcal{E}_p|_B \colon \mathcal{F}_p|_B \to [0,\infty)$ by $\mathcal{E}_p|_B(u) \coloneqq \mathcal{E}_p(h_B^{\mathcal{E}_p}[u])$ for $u \in \mathcal{F}_p|_B$. Then $(\mathcal{E}_p|_B, \mathcal{F}_p|_B)$ is a p-resistance form on B. Furthermore, $R_{\mathcal{E}_p|_B} = R_{\mathcal{E}_p}|_{B\times B}$ and*

$$\mathcal{E}_p|_B(u; v) = \mathcal{E}_p\bigl(h_B^{\mathcal{E}_p}[u]; h_B^{\mathcal{E}_p}[v]\bigr) \quad \textit{for any } u, v \in \mathcal{F}_p|_B.$$

Now let us state results on behaviors of $\mathcal{E}_p$-harmonic functions. We start with a *comparison principle* for $\mathcal{E}_p$-harmonic functions. Because of the *nonlinearity* of the operator $h_B^{\mathcal{E}_p}$ in Proposition 1.8, a *maximum principle* does not imply a comparison principle unlike the case $p = 2$. Fortunately, by virtue of Proposition 1.8 and the strong subadditivity in Proposition 1.2, we can establish a *weak comparison principle* for $\mathcal{E}_p$-harmonic functions in the following formulation (see Proposition 6.26).

$$\text{If } \emptyset \neq B \subseteq K \text{ and } u, v \in \mathcal{F}_p|_B \text{ satisfy } u \leq v \text{ on } B, \text{ then } h_B^{\mathcal{E}_p}[u] \leq h_B^{\mathcal{E}_p}[v]. \tag{1.10}$$

We also show a a stronger formulation of the weak comparison principle above under suitable assumptions; see Proposition 6.30. Next we discuss a (scale-invariant) *elliptic Harnack inequality* for non-negative $\mathcal{E}_p$-harmonic functions. In the case $p = 2$, one can show a Harnack-type estimate for $\mathcal{E}_2$-harmonic functions by using the maximum principle (see [Kig01, Proposition 3.2.7]). We can not follow this approach to get a Harnack-type inequality for $\mathcal{E}_p$-harmonic functions because of an issue due to the nonlinearity. However, by employing a similar approach as in [Cap07], we can show the following elliptic Harnack inequality under some extra assumptions including the existence of nice p-energy measures (see Theorem 6.37 for the precise statement); there exists a constant $C \in (0,\infty)$ such that for any $(x, s) \in K \times (0,\infty)$ and any $h \in \mathcal{F}_p$ which is $\mathcal{E}_p$-harmonic on $B_{\widehat{R}_p}(x, 2s)$ and $u \geq 0$, where $\widehat{R}_p \coloneqq R_{\mathcal{E}_p}^{1/(p-1)}$, it holds that

$$\sup_{B_{\widehat{R}_p}(x,s)} h \leq C \inf_{B_{\widehat{R}_p}(x,s)} h, \tag{1.11}$$

which implies a local Hölder continuity of h. Regarding continuity estimates for $\mathcal{E}_p$-harmonic functions, we also obtain the following sharp Hölder regularity estimate, which is a key ingredient of the proof of Theorem 1.6.

Theorem 1.10 (Theorem 6.31)**.** *Let $(\mathcal{E}_p, \mathcal{F}_p)$ be a p-resistance form on K and let B be a non-empty subset of K. We define $B^{\mathcal{F}_p} \coloneqq \bigcap_{u\in\mathcal{F}_p; u|_B=0} u^{-1}(0)$ and, for $x \in K \setminus B^{\mathcal{F}_p}$,*

$$\widehat{R}_p(x, B) \coloneqq \left(\sup\left\{ \frac{|u(x)|^p}{\mathcal{E}_p(u)} \;\middle|\; u \in \mathcal{F}_p, u|_B = 0, u(x) \neq 0 \right\}\right)^{\frac{1}{p-1}}.$$

Assume that $h \in \mathcal{F}_p$ is $\mathcal{E}_p$-harmonic on $K\setminus B$ and $\sup_B |h| < \infty$. Then for any $x \in K\setminus B^{\mathcal{F}_p}$ and any $y \in K$,

$$|h(x) - h(y)| \leq \frac{\widehat{R}_p(x, y)}{\widehat{R}_p(x, B)} \sup_{x',y'\in B} |h(x') - h(y')|.$$

Next let us move to applications of such a general theory of p-resistance forms. In forthcoming papers [KS.b, KS.c], the authors will heavily use this theory to make some essential progress in the setting of post-critically finite self-similar structures. See [KS23+] for a survey of these results in the case of the Sierpiński gasket. Here we shall explain another application for strict estimates on *p-walk dimensions* of some special classes of fractals, namely generalized Sierpiński carpets and D-dimensional level-l Sierpiński gasket (see Figure 1.2). For such a nice self-similar set K, as shown in the previous studies, we can construct $\mathcal{E}_p$ so as to satisfy the following *self-similarity*: there exists $\sigma_p \in (0,\infty)$ (which we call the *weight* of $(\mathcal{E}_p,\mathcal{F}_p)$) such that

$$\mathcal{E}_p(u) = \sigma_p \sum_{i\in S} \mathcal{E}_p(u \circ F_i), \quad u \in \mathcal{F}_p, \tag{1.12}$$

where S is a finite set and $\{F_i\}_{i\in S}$ is a family of similitudes associated with K such that $K = \bigcup_{i\in S} F_i(K)$ and $|F_i(x) - F_i(y)| = r_* \,|x-y|$ for some $r_* \in (0,1)$. Then the p-walk dimension $d_{\mathrm{w},p}$ of K is defined by

$$d_{\mathrm{w},p} \coloneqq \frac{\log\bigl((\#S)\sigma_p\bigr)}{\log r_*^{-1}},$$

which coincides with the walk-dimension if $p=2$. As shown in [MS23+, Theorem 7.1], the value $d_{\mathrm{w},p}$ plays a role of the space-scaling exponent in the following sense:

$$\mathcal{E}_p(u) \asymp \limsup_{r\downarrow 0} \int_K \fint_{|x-y|<r} \frac{|u(x)-u(y)|^p}{r^{d_{\mathrm{w},p}}}\, m(dy)m(dx), \quad u \in \mathcal{F}_p,$$

where m is the $\log(\#S)/\log r_*^{-1}$-dimensional Hausdorff measure on (K,d) with $m(K)=1$. In the case $p=2$, the strict inequality $d_{\mathrm{w},2} > 2$ has been verified for many self-similar sets, which implies a number of anomalous features of the diffusion associated with $(\mathcal{E}_2,\mathcal{F}_2)$. See [Kaj23] and the references therein for further details. Compared with the case $p=2$, a class of self-similar sets where $d_{\mathrm{w},p} > p$ is shown is limited to the *planar* generalized Sierpiński carpets due to the lack of counterparts of many useful tools in the case $p=2$ (see [Shi24, Theorem 2.27]). As an application of the differentiability in (1.8), in Section 9, we will extend this result to *any* generalized Sierpiński carpet by following the argument in [Kaj23]. We also prove $d_{\mathrm{w},p} > p$ for any D-dimensional level-l Sierpiński gasket, where the argument in [Kaj23] does not work.

We would also like to mention a geometric role of σ_p appearing in (1.12). As done in [Kig20, Kig23], the constant σ_p is determined by seeking the behavior of *conductance constants* (see [Kig23, Definition 2.17]) on approximating graphs of K. (See Theorem 8.12 for details.) A remarkable fact is that the behavior of σ_p as a function in p is deeply related to the notion of *Ahlfors regular conformal dimension*; indeed, $\sigma_p > 1$ if and only if $p > \dim_{\mathrm{ARC}}(K)$ (see, e.g., [Kig20, Theorem 4.7.6]), where $\dim_{\mathrm{ARC}}(K)$ denotes the Ahlfors regular conformal dimension of K (see Definition 8.5-(4) for the definition of $\dim_{\mathrm{ARC}}(K)$). Therefore, knowing properties of the function $p \mapsto \sigma_p$ is very important to understand the Ahlfors regular conformal dimension and related geometric information. Nevertheless, we do not know anything other than the following:

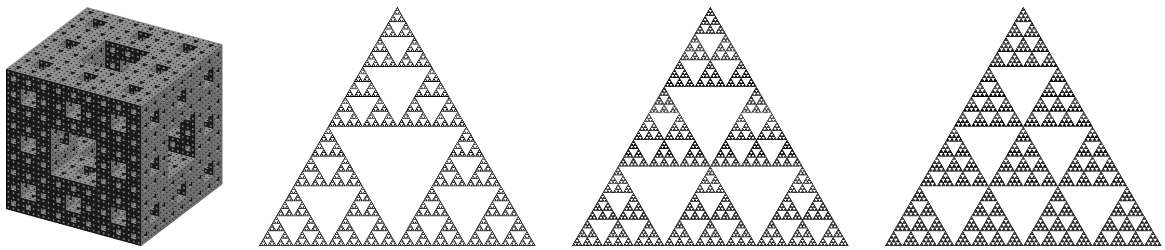

Figure 1.2: From the left, a non-planar generalized Sierpiński carpet (Menger Sponge) and 2-dimensional level-l Sierpiński gaskets ($l = 2, 3, 4$)

(Continuity; [Kig20, Proposition 4.7.5]) σ_p is continuous in p.
(Simple monotonicity; [Kig20, Proposition 4.7.5]) σ_p is non-decreasing in p.
(Hölder-type monotonicity; [Kig20, Lemma 4.7.4]) $d_{\mathrm{w},p}/p$ is non-increasing in p.
(Relation with $\dim_{\mathrm{ARC}}$; [Kig20, Theorem 4.7.6]) $\sigma_p > 1$ if and only if $p > \dim_{\mathrm{ARC}}(K)$.

As another application of our theory of p-resistance forms, we present the following new monotonicity behavior on σ_p (in a suitable general setting including all of the self-similar sets in Figure 1.2):

$$(\dim_{\mathrm{ARC}}(K), \infty) \ni p \mapsto \sigma_p^{1/(p-1)} \in (0, \infty) \text{ is non-decreasing in } p, \tag{1.13}$$

which is good evidence that properties of $p \mapsto \sigma_p^{1/(p-1)}$ are also important to deepen our understanding of $(\mathcal{E}_p, \mathcal{F}_p)$ and, possibly, of $\dim_{\mathrm{ARC}}(K)$.

Let us conclude the introduction by clarifying a difference between our theory and some related literatures [BBR24, Kuw24], where p-energy forms based on a (strongly local regular) symmetric Dirichlet form are considered. In the settings of [BBR24, Kuw24], the associated p-energy measure $\Gamma_p\langle u\rangle$ can be explicitly defined by using the "density" corresponding to $|\nabla u|$ without depending on p (see Example 4.2-(3)) whereas it is almost impossible to find a priori such a density on fractals. (We can naturally define p-energy measures by using (1.12). See Section 5 for details. See also [KS.a] for p-energy measure associated with Korevaar–Schoen p-energy forms.) In [KS.c], the authors will show that $\Gamma_p\langle u_p\rangle$ and $\Gamma_q\langle u_q\rangle$ are mutually singular with respect to each other for any $p, q \in (1, \infty)$ with $p \neq q$ and any $(u_p, u_q) \in \mathcal{F}_p \times \mathcal{F}_q$ for some post-critically finite self-similar sets by establishing the strict version of (1.13). This phenomenon on the singularity of energy measures *never* happens in the settings of [BBR24, Kuw24]. This point also motivates us to develop a general theory of p-energy forms in an abstract setting in order to deal with fractals.

This paper is organized as follows. In Section 2, we collect basic results on the generalized p-contraction property. In Section 3, we prove the differentiability (in Theorem 1.3) for p-energy forms satisfying p-Clarkson's inequality. Moreover, we will see that the (Fréchet) derivative in (1.8) gives a homeomorphism between $\mathcal{F}_p/\mathcal{E}_p^{-1}(0)$ and its dual. We also discuss regular and local properties of p-energy forms there. In Section 4, under the existence of p-energy measures, we discuss fundamental properties of them (Theorem 1.4 for example). We also formulate a chain rule for p-energy measures and observe some consequence of it. In Section 5, we give standard notations on self-similar structures, discuss the self-similarity of p-energy forms and see that we can associate self-similar p-energy measures to a given self-similar p-energy form. Section 6 is devoted to the study of fundamental nonlinear potential theory for p-resistance forms, most of which are mentioned in the introduction (see Theorems 1.6, 1.9, 1.10, Proposition 1.8, (1.10) and (1.11)). We further investigate the theory of p-resistance forms in the self-similar case in Section 7. In particular, we establish a Poincaré-type inequality in terms of self-similar p-energy measures under some geometric assumptions on the p-resistance metric. In Section 8, the generalized p-contraction property is verified for p-energy/p-resistance forms constructed in [Kig23, CGQ22]. More precisely, in Subsections 8.1 and 8.2, we recall the notion of p-conductively homogeneous compact metric space and the construction of $(\mathcal{E}_p, \mathcal{F}_p)$ due to [Kig23]. In Subsection 8.3, we focus our attempt on post-critically finite self-similar structures and show that *eigenforms* constructed in [CGQ22] turn out to be p-resistance forms. In Subsection 8.4, we review a sufficient condition for the existence of eigenforms on affine nested fractals. In Section 9, we prove $d_{\mathrm{w},p} > p$ for generalized Sierpiński carpets and D-dimensional level-l Sierpiński gasket by using properties of p-harmonic functions developed in Section 6.

Notation. Throughout this paper, we use the following notation and conventions.

(1) For $[0,\infty]$-valued quantities A and B, we write $A \lesssim B$ to mean that there exists an implicit constant $C \in (0,\infty)$ depending on some unimportant parameters such that $A \le CB$. We write $A \asymp B$ if $A \lesssim B$ and $B \lesssim A$.

(2) For a set A, we let $\#A \in \mathbb{N} \cup \{0,\infty\}$ denote the cardinality of A.

(3) We set $\sup\emptyset \coloneqq 0$ and $\inf\emptyset \coloneqq \infty$. We write $a \vee b \coloneqq \max\{a,b\}$, $a \wedge b \coloneqq \min\{a,b\}$ and $a^+ \coloneqq a \vee 0$ for $a, b \in [-\infty,\infty]$, and we use the same notation also for $[-\infty,\infty]$-valued functions and equivalence classes of them. All numerical functions in this paper are assumed to be $[-\infty,\infty]$-valued.

(4) Let $n \in \mathbb{N}$. For $x = (x_k)_{k=1}^n \in \mathbb{R}^n$, we set $\|x\|_{\ell_n^p} \coloneqq \|x\|_{\ell^p} \coloneqq (\sum_{k=1}^n |x_k|^p)^{1/p}$ for $p \in (0,\infty)$, $\|x\|_{\ell_n^\infty} \coloneqq \|x\|_{\ell^\infty} \coloneqq \max_{1\le k\le n} |x_k|$ and $\|x\| \coloneqq \|x\|_{\ell^2}$. For $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ which is differentiable on $\mathbb{R}^n$ and for $k \in \{1,\dots,n\}$, its first-order partial derivative in the k-th coordinate is denoted by $\partial_k\Phi$ and its gradient is denoted by $\nabla\Phi \coloneqq (\partial_k\Phi)_{k=1}^n$.

(5) Let X be a non-empty set. We define $\mathrm{id}_X\colon X \to X$ by $\mathrm{id}_X(x) \coloneqq x$, $\mathbb{1}_A = \mathbb{1}_A^X \in \mathbb{R}^X$ for $A \subseteq X$ by $\mathbb{1}_A(x) \coloneqq \mathbb{1}_A^X(x) \coloneqq \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$ and set $\|u\|_{\sup} \coloneqq \|u\|_{\sup,X} \coloneqq \sup_{x\in X} |u(x)|$ for $u\colon X \to [-\infty,\infty]$. Also, set $\operatorname{osc}_X[u] \coloneqq \sup_{x,y\in X} |u(x) - u(y)|$ for $u\colon X \to \mathbb{R}$ with $\|u\|_{\sup} < \infty$.

(6) Let X be a topological space. The Borel σ-algebra of X is denoted by $\mathcal{B}(X)$, the closure of $A \subseteq X$ in X by $\overline{A}^X$, and we say that $A \subseteq X$ is *relatively compact* in X if and only if $\overline{A}^X$ is compact. We set $C(X) \coloneqq \{u \in \mathbb{R}^X \mid u \text{ is continuous}\}$, $\operatorname{supp}_X[u] \coloneqq \overline{X \setminus u^{-1}(0)}^X$ for $u \in C(X)$, $C_b(X) \coloneqq \{u \in C(X) \mid \|u\|_{\sup} < \infty\}$, and $C_c(X) \coloneqq \{u \in C(X) \mid \operatorname{supp}_X[u] \text{ is compact}\}$.

(7) Let X be a topological space having a countable open base. For a Borel measure m

on X and a Borel measurable function $f\colon X \to [-\infty,\infty]$ or an m-equivalence class f of such functions, we let $\mathrm{supp}_m[f]$ denote the support of the measure $|f|\,dm$, that is, the smallest closed subset F of X such that $\int_{X\setminus F}|f|\,dm = 0$.

(8) Let (X,d) be a metric space. We set $B_d(x,r) \coloneqq \{y \in X \mid d(x,y) < r\}$ for $(x,r) \in X \times (0,\infty)$ and $\mathrm{dist}_d(A,B) \coloneqq \inf\{d(x,y) \mid x \in A, y \in B\}$ for subsets A, B of X.

(9) Let $(X,\mathcal{B},m)$ be a measure space. We set $⨍_A f\,dm \coloneqq \frac{1}{m(A)}\int_A f\,dm$ for $f \in L^1(X,m)$ and $A \in \mathcal{B}$ with $m(A) \in (0,\infty)$, and set $m|_A \coloneqq m|_{\mathcal{B}|_A}$ for $A \in \mathcal{B}$, where $\mathcal{B}|_A \coloneqq \{B \cap A \mid B \in \mathcal{B}\}$.

2 The generalized p-contraction property

In this section, we will introduce the generalized p-contraction property and establish basic results on these properties. Throughout this section, we fix $p \in (1,\infty)$, a measure space $(X,\mathcal{B},m)$, a linear subspace $\mathcal{F}$ of $L^0(X,m) \coloneqq L^0(X,\mathcal{B},m)$, where

$$L^0(X,\mathcal{B},m) \coloneqq \{\text{the } m\text{-equivalence class of } f \mid f\colon X \to \mathbb{R},\ f \text{ is } \mathcal{B}\text{-measurable}\},$$

and a p-homogeneous map $\mathcal{E}\colon \mathcal{F} \to [0,\infty)$, i.e., $\mathcal{E}(au) = |a|^p\,\mathcal{E}(u)$ for any $(a,u) \in \mathbb{R} \times \mathcal{F}$. (The pair $(\mathcal{B},m)$ is arbitrary. In the case where $\mathcal{B} = 2^X$ and m is the counting measure on X, we have $L^0(X,\mathcal{B},m) = \mathbb{R}^X$.)

Definition 2.1 (Generalized p-contraction property)**.** The pair $(\mathcal{E},\mathcal{F})$ is said to satisfy the *generalized p-contraction property*, $(\mathrm{GC})_p$ for short, if and only if the following hold: if $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,p]$, $q_2 \in [p,\infty]$ and $T = (T_1,\dots,T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy

$$T(0) = 0 \quad \text{and} \quad \|T(x) - T(y)\|_{\ell^{q_2}} \le \|x - y\|_{\ell^{q_1}} \quad \text{for any } x, y \in \mathbb{R}^{n_1}, \tag{2.1}$$

then for any $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in \mathcal{F}^{n_1}$ we have

$$T(\boldsymbol{u}) \in \mathcal{F}^{n_2} \quad \text{and} \quad \left\|\left(\mathcal{E}(T_l(\boldsymbol{u}))^{1/p}\right)_{l=1}^{n_2}\right\|_{\ell^{q_2}} \le \left\|\left(\mathcal{E}(u_k)^{1/p}\right)_{k=1}^{n_1}\right\|_{\ell^{q_1}}. \tag{GC$_p$}$$

The next proposition is a collection of useful inequalities included in $(\mathrm{GC})_p$.

Proposition 2.2. *Let $\varphi \in C(\mathbb{R})$ satisfy $\varphi(0) = 0$[4] and $|\varphi(t) - \varphi(s)| \le |t - s|$ for any $s, t \in \mathbb{R}$. Suppose that $(\mathcal{E},\mathcal{F})$ satisfies $(\mathrm{GC})_p$.*

(a) *$T(x,y) \coloneqq x + y$, $x, y \in \mathbb{R}$, satisfies (2.1) with $(q_1,q_2,n_1,n_2) = (1,p,2,1)$. In particular, $\mathcal{E}^{1/p}$ is a seminorm on $\mathcal{F}$, and $\mathcal{E}$ is strictly convex on $\mathcal{F}/\mathcal{E}^{-1}(0)$, i.e., for any $\lambda \in (0,1)$, any $f, g \in \mathcal{F}$ with $\mathcal{E}(f) \wedge \mathcal{E}(g) > 0$ and $f - g \notin \mathcal{E}^{-1}(0)$,*

$$\mathcal{E}(\lambda f + (1-\lambda)g) < \lambda\mathcal{E}(f) + (1-\lambda)\mathcal{E}(g). \tag{2.2}$$

(b) *$T \coloneqq \varphi$ satisfies (2.1) with $(q_1,q_2,n_1,n_2) = (1,p,1,1)$. In particular,*

$$\varphi(u) \in \mathcal{F} \text{ and } \mathcal{E}(\varphi(u)) \le \mathcal{E}(u) \text{ for any } u \in \mathcal{F}. \tag{2.3}$$

[4]Note that $\varphi \circ f \in L^p(X,m)$ for any $f \in L^p(X,m)$ by this condition.

(c) *Assume that φ is non-decreasing. Define $T = (T_1, T_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$T_1(x_1, x_2) = x_1 - \varphi(x_1 - x_2) \quad \text{and} \quad T_2(x_1, x_2) = x_2 + \varphi(x_1 - x_2), \quad (x_1, x_2) \in \mathbb{R}^2.$$

Then T satisfies (2.1) with $(q_1, q_2, n_1, n_2) = (p, p, 2, 2)$. In particular,

$$\mathcal{E}\bigl(f - \varphi(f - g)\bigr) + \mathcal{E}\bigl(g + \varphi(f - g)\bigr) \le \mathcal{E}(f) + \mathcal{E}(g) \quad \text{for any } f, g \in \mathcal{F}. \tag{2.4}$$

Moreover, by considering the case $\varphi(x) = x \vee 0$, we have the following strong subadditivity*: $f \vee g, f \wedge g \in \mathcal{F}$ and*

$$\mathcal{E}(f \vee g) + \mathcal{E}(f \wedge g) \le \mathcal{E}(f) + \mathcal{E}(g). \tag{2.5}$$

(d) *For any $a_1, a_2 > 0$, define $T^{a_1,a_2}\colon \mathbb{R}^2 \to \mathbb{R}$ by*

$$T^{a_1,a_2}(x_1, x_2) := \bigl(\bigl[(-a_1) \vee a_2^{-1}x_1\bigr] \wedge a_1\bigr) \cdot \bigl(\bigl[(-a_2) \vee a_1^{-1}x_2\bigr] \wedge a_2\bigr), \quad (x_1, x_2) \in \mathbb{R}^2.$$

Then T^{a_1,a_2} satisfies (2.1) with $(q_1, q_2, n_1, n_2) = (1, p, 2, 1)$. In particular, for any $f, g \in \mathcal{F} \cap L^\infty(X, m)$ we have

$$f \cdot g \in \mathcal{F} \quad \text{and} \quad \mathcal{E}(f \cdot g)^{1/p} \le \|g\|_{L^\infty(X,m)} \mathcal{E}(f)^{1/p} + \|f\|_{L^\infty(X,m)} \mathcal{E}(g)^{1/p}. \tag{2.6}$$

(e) *Assume that $p \in (1, 2]$. Define $T = (T_1, T_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$T_1(x_1, x_2) = 2^{-(p-1)/p}(x_1 + x_2) \quad \text{and} \quad T_2(x_1, x_2) = 2^{-(p-1)/p}(x_1 - x_2), \quad (x_1, x_2) \in \mathbb{R}^2.$$

Then T satisfies (2.1) with $(q_1, q_2, n_1, n_2) = (p/(p-1), p, 2, 2)$. In particular, $(\mathcal{E}, \mathcal{F})$ satisfies the following p-Clarkson's inequalities:

$$\mathcal{E}(f + g) + \mathcal{E}(f - g) \ge 2\bigl(\mathcal{E}(f)^{1/(p-1)} + \mathcal{E}(g)^{1/(p-1)}\bigr)^{p-1}, \tag{2.7}$$
$$\mathcal{E}(f + g) + \mathcal{E}(f - g) \le 2\bigl(\mathcal{E}(f) + \mathcal{E}(g)\bigr). \tag{2.8}$$

(f) *Assume that $p \in [2, \infty)$. Define $T = (T_1, T_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$T_1(x_1, x_2) = 2^{-1/p}(x_1 + x_2) \quad \text{and} \quad T_2(x_1, x_2) = 2^{-1/p}(x_1 - x_2), \quad (x_1, x_2) \in \mathbb{R}^2.$$

Then T satisfies (2.1) with $(q_1, q_2, n_1, n_2) = (p, p/(p-1), 2, 2)$. In particular, $(\mathcal{E}, \mathcal{F})$ satisfies the following p-Clarkson's inequalities:

$$\mathcal{E}(f + g) + \mathcal{E}(f - g) \le 2\bigl(\mathcal{E}(f)^{1/(p-1)} + \mathcal{E}(g)^{1/(p-1)}\bigr)^{p-1}, \tag{2.9}$$
$$\mathcal{E}(f + g) + \mathcal{E}(f - g) \ge 2\bigl(\mathcal{E}(f) + \mathcal{E}(g)\bigr). \tag{2.10}$$

Remark 2.3. (1) The property (2.4) is inspired by the *nonlinear Dirichlet form theory* due to Cipriani and Grillo [CG03]. See [Cla23, Theorem 4.7] and the reference therein for further background.

(2) By using an elementary inequality $2^{q-1}(a^q + b^q) \le (a + b)^q$ for $q \in (0, 1]$ and $a, b \in [0, \infty)$, we easily see that the inequality (2.7) for $(\mathcal{E}, \mathcal{F})$ in the case $p \in (1, 2]$ implies (2.8). Similarly, by Hölder's inequality, the inequality (2.9) for $(\mathcal{E}, \mathcal{F})$ in the case $p \in [2, \infty)$ implies (2.10).

Proof. (a): It is obvious that $T(x,y) := x+y$ satisfies (2.1) with $(q_1,q_2,n_1,n_2) = (1,p,2,1)$ and hence the triangle inequality for $\mathcal{E}^{1/p}$ holds. Since $(0,\infty) \ni x \mapsto x^p$ is strictly convex, for any $\lambda \in (0,1)$ and any $f,g \in \mathcal{F}$ with $\mathcal{E}(f) \wedge \mathcal{E}(g) \wedge \mathcal{E}(f-g) > 0$,

$$\mathcal{E}(\lambda f + (1-\lambda)g) \leq \bigl(\lambda\mathcal{E}(f)^{1/p} + (1-\lambda)\mathcal{E}(g)^{1/p}\bigr)^p < \lambda\mathcal{E}(f) + (1-\lambda)\mathcal{E}(g),$$

where we used the triangle inequality for $\mathcal{E}^{1/p}$ in the first inequality.

(b): This is obvious.

(c): Let $x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2$. For simplicity, set $z_i := x_i - y_i$ and $A := \varphi(x_1-x_2) - \varphi(y_1-y_2)$. Then $\|T(x)-T(y)\|_{\ell^p} \leq \|x-y\|_{\ell^p}$ ie equivalent to

$$|z_1 - A|^p + |z_2 + A|^p \leq |z_1|^p + |z_2|^p\,, \tag{2.11}$$

so we will show (2.11). Note that $|A| \leq |z_1 - z_2|$ since φ is 1-Lipschitz. The desired estimate (2.11) is evident when $z_1 = z_2$, so we consider the case $z_1 \neq z_2$. Suppose that $z_1 > z_2$ because the remaining case $z_1 < z_2$ is similar. Then $(x_1-x_2)-(y_1-y_2) = z_1 - z_2 > 0$ and thus $0 \leq A \leq z_1 - z_2$. Set $\psi_p(t) := |t|^p$ $(t \in \mathbb{R})$ for brevity. If $0 \leq A < \frac{z_1-z_2}{2}$, then $z_2 \leq z_2 + A < z_1 - A \leq z_1$ and we see that

$$\begin{aligned}|z_1 - A|^p + |z_2 + A|^p - |z_1|^p - |z_2|^p &= \int_{z_2}^{z_2+A} \psi_p'(t)\,dt - \int_{z_1-A}^{z_1} \psi_p'(t)\,dt \\ &\leq A\bigl(\psi_p'(z_2+A) - \psi_p'(z_1-A)\bigr) \leq 0.\end{aligned}$$

If $A \geq \frac{z_1-z_2}{2}$, then $z_2 \leq z_1 - A \leq z_2 + A \leq z_1$ and thus

$$\begin{aligned}|z_1 - A|^p + |z_2 + A|^p - |z_1|^p - |z_2|^p &= \int_{z_2}^{z_1-A} \psi_p'(t)\,dt - \int_{z_2+A}^{z_1} \psi_p'(t)\,dt \\ &\leq (z_1 - z_2 - A)\bigl(\psi_p'(z_1-A) - \psi_p'(z_2+A)\bigr) \leq 0,\end{aligned}$$

which proves (2.11). The case $\varphi(x) = x^+$ immediately implies (2.5).

(d): For any $a_1, a_2 > 0$ and $(x_1,x_2),(y_1,y_2) \in \mathbb{R}^2$, we see that

$$\begin{aligned}&|T^{a_1,a_2}(x_1,x_2) - T^{a_1,a_2}(x_1,x_2)| \\ &\leq \bigl|(-a_1) \vee a_2^{-1}x_1 \wedge a_1\bigr| \bigl|\bigl((-a_2) \vee a_1^{-1}x_2 \wedge a_2\bigr) - \bigl((-a_2) \vee a_1^{-1}y_2 \wedge a_2\bigr)\bigr| \\ &\quad + \bigl|(-a_2) \vee a_1^{-1}y_2 \wedge a_2\bigr| \bigl|\bigl((-a_1) \vee a_2^{-1}x_1 \wedge a_1\bigr) - \bigl((-a_1) \vee a_2^{-1}y_1 \wedge a_1\bigr)\bigr| \\ &\leq a_1\left|a_1^{-1}x_2 - a_1^{-1}y_2\right| + a_2\left|a_2^{-1}x_1 - a_2^{-1}y_1\right| = |x_1 - y_1| + |x_2 - y_2|\,,\end{aligned}$$

whence T^{a_1,a_2} satisfies (2.1). We get (2.6) by applying (GC)_p with $u_1 = \|g\|_{L^\infty(X,m)} f$, $u_2 = \|f\|_{L^\infty(X,m)} g$, $a_1 = \|f\|_{L^\infty(X,m)}$, $a_2 = \|g\|_{L^\infty(X,m)}$.

(e),(f): These statements follow from p-Clarkson's inequalities for the ℓ^p-norm (see, e.g., [Cla36, Theorem 2]). □

The following corollary is easily implied by Proposition 2.2-(b),(d).

Corollary 2.4. *Assume that $(\mathcal{E},\mathcal{F})$ satisfies (GC)_p.*

(a) *Let* $u \in \mathcal{F} \cap L^{\infty}(X,m)$ *and let* $\Phi \in C^{1}(\mathbb{R})$ *satisfy* $\Phi(0)=0$. *Then*

$$\Phi(u) \in \mathcal{F} \quad \text{and} \quad \mathcal{E}(\Phi(u)) \le \sup\bigl\{|\Phi'(t)|^{p} \bigm| t \in \mathbb{R}, |t| \le \|u\|_{L^{\infty}(X,m)}\bigr\}\mathcal{E}(u). \tag{2.12}$$

(b) *Let* $\delta, M \in (0,\infty)$ *and let* $f, g \in \mathcal{F}$ *satisfy* $f \ge 0$, $g \ge 0$, $f \le M$ *and* $(f+g)|_{\{f \neq 0\}} \ge \delta$. *Then there exists* $C \in (0,\infty)$ *depending only on* p, δ, M *such that*

$$\frac{f}{f+g} \in \mathcal{F} \quad \text{and} \quad \mathcal{E}\left(\frac{f}{f+g}\right) \le C\bigl(\mathcal{E}(f)+\mathcal{E}(g)\bigr). \tag{2.13}$$

(c) *Let* $n \in \mathbb{N}$, $v \in \mathcal{F}$ *and* $\boldsymbol{u} = (u_1, \dots, u_n) \in L^{0}(X,m)^{n}$. *If there exist* $q \in [1,p]$ *and* m*-versions of* $v, \boldsymbol{u}$ *such that* $|v(x)| \le \|\boldsymbol{u}(x)\|_{\ell^{q}}$ *and* $|v(x)-v(y)| \le \|\boldsymbol{u}(x)-\boldsymbol{u}(y)\|_{\ell^{q}}$ *for any* $x, y \in X$, *then* $\boldsymbol{u} \in \mathcal{F}^{n}$ *and* $\mathcal{E}(v) \le \bigl\|\bigl(\mathcal{E}(v_k)^{1/p}\bigr)_{k=1}^{n}\bigr\|_{\ell^{q}}$.

Proof. The statement (a) is immediate from Proposition 2.2-(b).

(b): We follow [MS23+, Proposition 6.25 (ii)]. Let $\varphi \in C(\mathbb{R})$ be a Lipschitz map such that $\varphi(x) = x^{-1}$ for $x \ge \delta$ and $\sup_{x \neq y \in \mathbb{R}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|} \le C'$ for some constant C' depending only on δ. Since $f \cdot \varphi(f+g) = \frac{f}{f+g}$, we get (2.13) by using (2.3) and (2.6).

(c): The proof below is similar to [MR, Corollary I.4.13]. Fix m-versions of $v, \boldsymbol{u}$ satisfying $|v(x)| \le \|\boldsymbol{u}(x)\|_{\ell^{q}}$ and $|v(x)-v(y)| \le \|\boldsymbol{u}(x)-\boldsymbol{u}(y)\|_{\ell^{q}}$ for any $x, y \in X$. Set $\boldsymbol{u}(X) \coloneqq u_1(X) \times \cdots \times u_n(X) \subseteq \mathbb{R}^{n}$. We define $T_0 \colon \boldsymbol{u}(X) \to \mathbb{R}$ by setting $T_0(0) \coloneqq 0$ and $T_0(\boldsymbol{z}) \coloneqq v(x)$ for each $\boldsymbol{z} \in \boldsymbol{u}(X)$, where $x \in X$ satisfies $\boldsymbol{z} = \boldsymbol{u}(x)$. This map T_0 is well-defined since $v(x) = 0$ for any $x \in X$ with $\boldsymbol{u}(x) = 0$ and $|v(x)-v(y)| \le \|\boldsymbol{u}(x)-\boldsymbol{u}(y)\|_{\ell^{q}} = 0$ for any $x, y \in X$ with $\boldsymbol{u}(x) = \boldsymbol{u}(y) \in \boldsymbol{u}(X)$. In addition, we easily see that $|T_0(\boldsymbol{z}_1) - T_0(\boldsymbol{z}_2)| \le \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_{\ell^{q}}$ for any $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \boldsymbol{u}(X) \cup \{0\}$, i.e., $T_0 \colon (\boldsymbol{u}(X) \cup \{0\}, \|\cdot\|_{\ell^{q}}) \to \mathbb{R}$ is 1-Lipschitz. Noting that $(\mathbb{R}^{n}, \|\cdot\|_{\ell^{q}})$ is a metric space since $q \ge 1$, we can get a 1-Lipschitz map $T \colon (\mathbb{R}^{n}, \|\cdot\|_{\ell^{q}}) \to \mathbb{R}$ satisfying $T(\boldsymbol{z}) = T_0(\boldsymbol{z})$ for any $\boldsymbol{z} \in \boldsymbol{u}(X) \cup \{0\}$ by applying the McShane–Whitney extension lemma (see, e.g., [HKST, p. 99]). Since T satisfies (2.1) with $(q_1, q_2, n_1, n_2) = (q, p, n, 1)$ and $\mathcal{E}(T(\boldsymbol{u})) = \mathcal{E}(v)$, we obtain the desired statement by $(\mathrm{GC})_p$. □

We also notice that $(\mathrm{GC})_p$ includes a new variant of p-Clarkson's inequality in the case $p \in [2,\infty)$, which we call improved p-Clarkson's inequality. This result is not used in the paper, but we record it for potential future applications.

Proposition 2.5 (Improved p-Clarkson's inequality)**.** *Define* $\phi_p \colon (0,\infty) \to (0,\infty)$ *and* $T^{s} = (T_1^{s}, T_2^{s}) \colon \mathbb{R}^{2} \to \mathbb{R}^{2}$, $s \in (0,\infty)$, *by*

$$\psi_p(s) \coloneqq (1+s)^{p-1} + \operatorname{sgn}(1-s)\,|1-s|^{p-1}, \quad s > 0. \tag{2.14}$$

(a) *Assume that* $p \in (1,2]$. *For* $s \in (0,\infty)$, *define* $T^{s} = (T_1^{s}, T_2^{s}) \colon \mathbb{R}^{2} \to \mathbb{R}^{2}$ *by*

$$T_1^{s}(x_1,x_2) \coloneqq 2^{-1}\psi_p(s)^{1/p}(x_1+x_2), \quad T_2^{s}(x_1,x_2) \coloneqq 2^{-1}\psi_p(s^{-1})^{1/p}(x_1-x_2).$$

Then T^{s} *satisfies* (2.1) *with* $(q_1,q_2,n_1,n_2) = (p,p,2,2)$ *for any* $s \in (0,\infty)$. *If* $(\mathcal{E},\mathcal{F})$ *satisfies* $(\mathrm{GC})_p$, *then*

$$\sup_{s>0}\bigl\{\psi_p(s)\mathcal{E}(f) + \psi_p(s^{-1})\mathcal{E}(g)\bigr\} \le \mathcal{E}(f+g) + \mathcal{E}(f-g) \quad \text{for any } f, g \in \mathcal{F}. \tag{2.15}$$

(b) *If $p \in (1, 2]$ and $\mathcal{E}$ satisfies (2.15), then p-Clarkson's inequality, (2.7), for $\mathcal{E}$ holds.*

(c) *Assume that $p \in [2, \infty)$. For $s \in (0, \infty)$, define $T^s = (T_1^s, T_2^s) \colon \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$T_1^s(x_1, x_2) := \psi_p(s)^{-1/p}x_1 + \psi_p(s^{-1})^{-1/p}x_2, \quad T_2^s(x_1, x_2) := \psi_p(s)^{-1/p}x_1 - \psi_p(s^{-1})^{-1/p}x_2.$$

Then T^s satisfies (2.1) with $(q_1, q_2, n_1, n_2) = (p, p, 2, 2)$ for any $s \in (0, \infty)$. If $p \in [2, \infty)$ and $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{GC})_p$, then

$$\mathcal{E}(f + g) + \mathcal{E}(f - g) \le \inf_{s>0}\bigl\{\psi_p(s)\mathcal{E}(f) + \psi_p(s^{-1})\mathcal{E}(g)\bigr\} \quad \text{for any } f, g \in \mathcal{F}. \tag{2.16}$$

(d) *If $\mathcal{E}$ satisfies (2.16), then p-Clarkson's inequality, (2.9), for $\mathcal{E}$ holds.*

Proof. We first recall a key result from [BCL94, Lemma 4]: for any $x, y \in \mathbb{R}$,

$$|x + y|^p + |x - y|^p = \begin{cases} \sup_{s>0}\bigl\{\psi_p(s)\,|x|^p + \psi_p(s^{-1})\,|y|^p\bigr\} & \text{if } p \in [2, \infty), \\ \inf_{s>0}\bigl\{\psi_p(s)\,|x|^p + \psi_p(s^{-1})\,|y|^p\bigr\} & \text{if } p \in [2, \infty). \end{cases} \tag{2.17}$$

(a): By considering $x+y, x-y$ in (2.17) instead of x, y, we have that for any $s \in (0, \infty)$,

$$2^{-p}\phi_p(s)\,|x + y|^p + 2^{-p}\phi_p(s^{-1})\,|x - y|^p \le |x|^p + |y|^p\,,$$

which means that T^s satisfies (2.1) with $(q_1, q_2, n_1, n_2) = (p, p, 2, 2)$. Since $s \in (0, \infty)$ is arbitrary, we obtain (2.15).

(b): Let $f, g \in \mathcal{F}$ with $\mathcal{E}(f) \wedge \mathcal{E}(g) > 0$, set $a := \mathcal{E}(f)^{1/(p-1)}$ and $b := \mathcal{E}(g)^{1/(p-1)}$. Then,

$$\sup_{s>0}\bigl\{\psi_p(s)\mathcal{E}(f) + \psi_p(s^{-1})\mathcal{E}(g)\bigr\} \ge \psi_p(b/a)a^{p-1} + \psi_p(a/b)b^{p-1} = 2(a + b)^{p-1},$$

which together with (2.15) yields (2.7).

(c): For any $s \in (0, \infty)$, we immediately see from (2.17) that T^s satisfies (2.1). Since $s \in (0, \infty)$ is arbitrary, we obtain (2.16).

(d): Let $f, g \in \mathcal{F}$ with $\mathcal{E}(f) \wedge \mathcal{E}(g) > 0$, set $a := \mathcal{E}(f)^{1/(p-1)}$ and $b := \mathcal{E}(g)^{1/(p-1)}$. Then,

$$\inf_{s>0}\bigl\{\psi_p(s)\mathcal{E}(f) + \psi_p(s^{-1})\mathcal{E}(g)\bigr\} \le \psi_p(b/a)a^{p-1} + \psi_p(a/b)b^{p-1} = 2(a + b)^{p-1},$$

which together with (2.16) yields (2.9). □

The property $(\mathrm{GC})_p$ is stable under taking "limits" and some algebraic operations like summations. To state precise results, we recall the following definition on convergences of functionals.

Definition 2.6 ([Dal, Definition 4.1 and Proposition 8.1])**.** Let $\mathcal{X}$ be a topological space, let $F \colon \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$ and let $\{F_n \colon \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}\}_{n\in\mathbb{N}}$.

(1) The sequence $\{F_n\}_{n\in\mathbb{N}}$ is said to converge pointwise to F if and only if $\lim_{n\to\infty} F_n(x) = F(x)$ for any $x \in \mathcal{X}$.

(2) Suppose that $\mathcal{X}$ is a first-countable topological space. The sequence $\{F_n\}_{n\in\mathbb{N}}$ is said to Γ-converge to F (with respect to the topology of $\mathcal{X}$) if and only if the following conditions hold for any $x \in \mathcal{X}$:
   (i) If $x_n \to x$ in $\mathcal{X}$, then $F(x) \leq \liminf_{n\to\infty} F_n(x_n)$.
   (ii) There exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}$ such that

$$x_n \to x \text{ in } \mathcal{X} \quad \text{and} \quad \limsup_{n\to\infty} F_n(x_n) \leq F(x). \tag{2.18}$$

   A sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying (2.18) is called a *recovery sequence of* $\{F_n\}_{n\in\mathbb{N}}$ *at* x.

We also need the following reverse Minkowski inequality (see, e.g., [AF, Theorem 2.12]).

Proposition 2.7 (Reverse Minkowski inequality)**.** *Let* $(Y, \mathcal{A}, \mu)$ *be a measure space*[5] *and let* $r \in (0, 1]$*. Then for any* $\mathcal{A}$*-measurable functions* $f, g \colon Y \to [0, \infty]$,

$$\left(\int_Y f^r\, d\mu\right)^{1/r} + \left(\int_Y g^r\, d\mu\right)^{1/r} \leq \left(\int_Y (f+g)^r\, d\mu\right)^{1/r}. \tag{2.19}$$

In the following definition, we introduce the set of p-homogeneous functional on $\mathcal{F}$ which satisfies (GC)$_p$.

Definition 2.8. Recall that $\mathcal{F}$ is a linear subspace of $L^0(X, m)$. Define

$$\mathcal{U}_p^{\mathrm{GC}}(\mathcal{F}) \coloneqq \mathcal{U}_p^{\mathrm{GC}} \coloneqq \{\mathcal{E}' \colon \mathcal{F} \to [0, \infty) \mid \mathcal{E}' \text{ is } p\text{-homogeneous, } (\mathcal{E}', \mathcal{F}) \text{ satisfies } (\mathrm{GC})_p\}.$$

Now we can state the desired *stability* of (GC)$_p$.

Proposition 2.9. (a) $a_1\mathcal{E}^{(1)} + a_2\mathcal{E}^{(2)} \in \mathcal{U}_p^{\mathrm{GC}}$ *for any* $\mathcal{E}^{(1)}, \mathcal{E}^{(2)} \in \mathcal{U}_p^{\mathrm{GC}}$ *and any* $a_1, a_2 \in [0, \infty)$.
(b) *Let* $\left\{\mathcal{E}^{(n)} \in \mathcal{U}_p^{\mathrm{GC}}\right\}_{n\in\mathbb{N}}$ *and let* $\mathcal{E}^{(\infty)} \colon \mathcal{F} \to [0, \infty)$*. If* $\{\mathcal{E}^{(n)}\}_{n\in\mathbb{N}}$ *converges pointwise to* $\mathcal{E}^{(\infty)}$*, then* $\mathcal{E}^{(\infty)} \in \mathcal{U}_p^{\mathrm{GC}}$.
(c) *Suppose that* $\mathcal{F} \subseteq L^p(X, m)$ *and let us regard* $\mathcal{F}$ *as a topological space equipped with the topology of* $L^p(X, m)$*. Let* $\left\{\mathcal{E}^{(n)} \in \mathcal{U}_p^{\mathrm{GC}}\right\}_{n\in\mathbb{N}}$ *and let* $\mathcal{E}^{(\infty)} \colon \mathcal{F} \to [0, \infty)$*. If* $\{\mathcal{E}^{(n)}\}_{n\in\mathbb{N}}$ Γ*-converges to* $\mathcal{E}^{(\infty)}$*, then* $\mathcal{E}^{(\infty)} \in \mathcal{U}_p^{\mathrm{GC}}$.

Proof. The statement (b) is trivial, so we will show (a) and (c). Throughout this proof, we fix $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0, p]$, $q_2 \in [p, \infty]$ and $T = (T_1, \dots, T_{n_2}) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfying (2.1).

(a): Let $\mathcal{E}^{(1)}, \mathcal{E}^{(2)} \in \mathcal{U}_p^{\mathrm{GC}}$. Then $a\mathcal{E}^{(1)} \in \mathcal{U}_p^{\mathrm{GC}}$ is evident for any $a \in [0, \infty)$. Set $E(f) \coloneqq \mathcal{E}^{(1)}(f) + \mathcal{E}^{(2)}(f)$, $f \in \mathcal{F}$, and let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}^{n_1}$. It suffices to prove

[5] In the book [AF], the reverse Minkowski inequality is stated and proved only for the L^r-space over non-empty open subsets of the Euclidean space equipped with the Lebesgue measure, but the same proof works for any measure space.

$\left\|\left(E(T_l(\boldsymbol{u}))^{1/p}\right)_{l=1}^{n_2}\right\|_{\ell^{q_2}} \leq \left\|\left(E(u_k)^{1/p}\right)_{k=1}^{n_1}\right\|_{\ell^{q_1}}$. For simplicity, we consider the case $q_2 < \infty$. (The case $q_2 = \infty$ is similar.) Then we have

$$\begin{aligned}
&\sum_{l=1}^{n_2} E\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p} \\
&= \sum_{l=1}^{n_2}\Bigl[\mathcal{E}^{(1)}\bigl(T_l(\boldsymbol{u})\bigr) + \mathcal{E}^{(2)}\bigl(T_l(\boldsymbol{u})\bigr)\Bigr]^{q_2/p} \\
&\leq \left(\sum_{i\in\{1,2\}}\left[\sum_{l=1}^{n_2}\mathcal{E}^{(i)}\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p}\right]^{p/q_2}\right)^{q_2/p} \qquad \text{(by the triangle ineq. for } \|\cdot\|_{\ell^{q_2/p}}) \\
&\overset{(\mathrm{GC})_p}{\leq} \left(\left[\sum_{k=1}^{n_1}\mathcal{E}^{(1)}(u_k)^{q_1/p}\right]^{p/q_1} + \left[\sum_{k=1}^{n_1}\mathcal{E}^{(2)}(u_k)^{q_1/p}\right]^{p/q_1}\right)^{q_2/p} \\
&\overset{(2.19)}{\leq} \left(\sum_{k=1}^{n_1}\Bigl[\mathcal{E}^{(1)}(u_k) + \mathcal{E}^{(2)}(u_k)\Bigr]^{q_1/p}\right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}} = \left(\sum_{k=1}^{n_1} E(u_k)^{q_1/p}\right)^{q_2/q_1}, \qquad (2.20)
\end{aligned}$$

which implies $E \in \mathcal{U}_p^{\mathrm{GC}}$.

(c): Let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}^{n_1}$ and choose a recovery sequence $\{\boldsymbol{u}_n = (u_{1,n}, \dots, u_{n_1,n}) \in \mathcal{F}^{n_1}\}_{n\in\mathbb{N}}$ of $\{\mathcal{E}^{(n)}\}_{n\in\mathbb{N}}$ at $\boldsymbol{u}$. We first show tnat $\|T_l(\boldsymbol{u}) - T_l(\boldsymbol{u}_n)\|_{L^p(X,m)} \to 0$ as $n \to \infty$. Indeed, for any $\boldsymbol{v} = (v_1, \dots, v_{n_1})$ and any $\boldsymbol{z} = (z_1, \dots, z_{n_1}) \in L^p(X, m)^{n_1}$, we see that

$$\begin{aligned}
\max_{l\in\{1,\dots,n_2\}} \|T_l(\boldsymbol{v}) - T_l(\boldsymbol{z})\|_{L^p(X,m)}^p &\overset{(2.1)}{\leq} \int_X \|\boldsymbol{v}(x) - \boldsymbol{z}(x)\|_{\ell^{q_1}}^p \, m(dx) \\
&= \int_X \left(\sum_{k=1}^{n_1} |v_k(x) - z_k(x)|^{p\cdot\frac{q_1}{p}}\right)^{p/q_1} m(dx) \\
&\leq n_1^{(p-q_1)/q_1} \sum_{k=1}^{n_1} \|v_k - z_k\|_{L^p(X,m)}^p, \qquad (2.21)
\end{aligned}$$

where we used Hölder's inequality in the last line. Since $\max_k \|u_k - u_{k,n}\|_{L^p(X,m)} \to 0$ as $n \to \infty$, (2.21) implies the desired convergence $\|T_l(\boldsymbol{u}) - T_l(\boldsymbol{u}_n)\|_{L^p(X,m)} \to 0$.

Now we prove $(\mathrm{GC})_p$ for the Γ-limit $\mathcal{E}^{(\infty)}$ of $\{\mathcal{E}^{(n)}\}_{n\in\mathbb{N}}$ (with respect to the $L^p(X, m)$-topology). It is easy to see that $\mathcal{E}^{(\infty)}$ is p-homogeneous (see, e.g., [Dal, Proposition 11.6]). We suppose that $q_2 < \infty$ since the case $q_2 = \infty$ is similar. Then,

$$\begin{aligned}
\sum_{l=1}^{n_2} \mathcal{E}^{(\infty)}\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p} &\leq \sum_{l=1}^{n_2} \liminf_{n\to\infty} \mathcal{E}^{(n)}\bigl(T_l(\boldsymbol{u}_n)\bigr)^{q_2/p} \leq \liminf_{n\to\infty} \sum_{l=1}^{n_2} \mathcal{E}^{(n)}\bigl(T_l(\boldsymbol{u}_n)\bigr)^{q_2/p} \\
&\leq \liminf_{n\to\infty} \left(\sum_{k=1}^{n_1} \mathcal{E}^{(n)}(u_{k,n})^{q_1/p}\right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}} = \left(\sum_{k=1}^{n_1} \mathcal{E}^{(\infty)}(u_k)^{q_1/p}\right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}},
\end{aligned}$$

which proves $\mathcal{E}^{(\infty)} \in \mathcal{U}_p^{\mathrm{GC}}$. □

3 Differentiability of p-energy forms and related results

In this section, we show the existence of the derivative (1.8) for any p-energy form satisfying p-Clarkson's inequality, (2.7) or (2.9). As an application of our differentiability result, we will introduce a 'two-variable' version of a p-energy form and observe its fundamental properties.

Throughout this section, we fix $p \in (1, \infty)$, a measure space $(X, \mathcal{B}, m)$, and a *p-energy form* $(\mathcal{E}, \mathcal{F})$ on (X, m) in the following sense:

Definition 3.1 (p-Energy form)**.** Let $\mathcal{F}$ be a linear subspace of $L^0(X, m)$ and let $\mathcal{E}\colon \mathcal{F} \to [0, \infty)$. The pair $(\mathcal{E}, \mathcal{F})$ is said to be a *p-energy form* on (X, m) if $\mathcal{E}^{1/p}$ is a seminorm on $\mathcal{F}$.

Note that the same argument as in the proof of Proposition 2.2-(a) implies that $\mathcal{E}$ is strictly convex on $\mathcal{F}/\mathcal{E}^{-1}(0)$ (see (2.2)).

3.1 p-Clarkson's inequality and differentiability

In this section, we mainly deal with p-energy forms satisfying p-Clarkson's inequality in the following sense.

Definition 3.2 (p-Clarkson's inequality)**.** The pair $(\mathcal{E}, \mathcal{F})$ is said to satisfy *p-Clarkson's inequality*, $(\mathrm{Cla})_p$ for short, if and only if for any $f, g \in \mathcal{F}$,

$$\begin{cases} \mathcal{E}(f+g)^{1/(p-1)} + \mathcal{E}(f-g)^{1/(p-1)} \leq 2\bigl(\mathcal{E}(f) + \mathcal{E}(g)\bigr)^{1/(p-1)} & \text{if } p \in (1, 2], \\ \mathcal{E}(f+g) + \mathcal{E}(f-g) \leq 2\bigl(\mathcal{E}(f)^{1/(p-1)} + \mathcal{E}(g)^{1/(p-1)}\bigr)^{p-1} & \text{if } p \in [2, \infty). \end{cases} \tag{Cla_p}$$

To state a consequence of $(\mathrm{Cla})_p$ on the convexity of $\mathcal{E}^{1/p}$, let us recall the notion of uniform convexity. See, e.g., [Cla36, Definition 1]. (The notion of uniform convexity is usually defined for a Banach space in the literature. We present the definition for seminormed space because we are mainly interested in $(\mathcal{F}, \mathcal{E}^{1/p})$.)

Definition 3.3 (Uniformly convex seminormed spaces)**.** Let $(\mathcal{X}, |\cdot|)$ be a seminormed space. We say that $(\mathcal{X}, |\cdot|)$ is *uniformly convex* if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ with the property that $|f + g| \leq 2(1 - \delta)$ whenever $f, g \in \mathcal{X}$ satisfy $|f| = |g| = 1$ and $|f - g| > \varepsilon$.

It is well known that $(\mathrm{Cla})_p$ implies the uniform convexity as follows.

Proposition 3.4. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{Cla})_p$. Then $(\mathcal{F}, \mathcal{E}^{1/p})$ is uniformly convex.*

Proof. The same argument as in [Cla36, Proof of Corollary of Theorem 2] works. □

Moreover, $(\mathrm{Cla})_p$ provides us the following quantitative estimate for the central difference, which plays a central role in this section.

Proposition 3.5. *Assume that* $(\mathcal{E},\mathcal{F})$ *satisfies* $(\mathrm{Cla})_p$*. Then, for any* $f,g\in\mathcal{F}$*,*

$$\mathcal{E}(f+g)+\mathcal{E}(f-g)-2\mathcal{E}(f)\leq\begin{cases}2\mathcal{E}(g) & \text{if } p\in(1,2],\\ 2(p-1)\Bigl[\mathcal{E}(f)^{\frac{1}{p-1}}+\mathcal{E}(g)^{\frac{1}{p-1}}\Bigr]^{p-2}\mathcal{E}(g)^{\frac{1}{p-1}} & \text{if } p\in(2,\infty).\end{cases}\tag{3.1}$$

In particular, $\mathbb{R}\ni t\mapsto\mathcal{E}(f+tg)\in[0,\infty)$ *is differentiable and for any* $s\in\mathbb{R}$*,*

$$\lim_{\delta\downarrow 0}\sup_{g\in\mathcal{F};\mathcal{E}(g)\leq 1}\left|\frac{\mathcal{E}(f+(s+\delta)g)-\mathcal{E}(f+sg)}{\delta}-\frac{d}{dt}\mathcal{E}(f+tg)\biggr|_{t=s}\right|=0.\tag{3.2}$$

Proof. The desired inequality (3.1) in the case $p\in(1,2]$ is immediate from (2.8), so we suppose that $p\in(2,\infty)$. Let $f,g\in\mathcal{F}$, set $a:=\mathcal{E}(f)^{1/(p-1)}$ and $b:=\mathcal{E}(g)^{1/(p-1)}$. Then we have (3.1) since $(\mathrm{Cla})_p$ implies that

$$\mathcal{E}(f+g)+\mathcal{E}(f-g)-2\mathcal{E}(f)\leq 2\bigl((a+b)^{p-1}-a^{p-1}\bigr)=2(p-1)\int_a^{a+b}s^{p-2}\,ds\leq 2(p-1)(a+b)^{p-2}b.$$

Next we show that for any $t\in\mathbb{R}$,

$$\lim_{\delta\downarrow 0}\sup_{g\in\mathcal{F};\mathcal{E}(g)\leq 1}\frac{\mathcal{E}\bigl(f+(t+\delta)g\bigr)+\mathcal{E}\bigl(f+(t-\delta)g\bigr)-2\mathcal{E}(f+tg)}{\delta}=0,\tag{3.3}$$

Let $t\in\mathbb{R}$, $\delta\in(0,\infty)$ and set

$$D_{t,\delta}(f;g):=\mathcal{E}\bigl(f+(t+\delta)g\bigr)+\mathcal{E}\bigl(f+(t-\delta)g\bigr)-2\mathcal{E}(f+tg).\tag{3.4}$$

By (3.1), we have

$$D_{t,\delta}(f;g)\leq\begin{cases}2\delta^p\mathcal{E}(g) & \text{if } p\in(1,2],\\ 2(p-1)\delta^{p/(p-1)}\Bigl[\mathcal{E}(f+tg)^{\frac{1}{p-1}}+\mathcal{E}(\delta g)^{\frac{1}{p-1}}\Bigr]^{p-2}\mathcal{E}(g)^{\frac{1}{p-1}} & \text{if } p\in(2,\infty).\end{cases}$$

Hence we get

$$\sup_{g\in\mathcal{F};\mathcal{E}(g)\leq 1}\frac{D_{t,\delta}(f;g)}{\delta}\leq\begin{cases}2\delta^{p-1} & \text{if } p\in(1,2],\\ 2(p-1)\delta^{1/(p-1)}\Bigl[\bigl(\mathcal{E}(f)^{1/p}+t\bigr)^{\frac{p}{p-1}}+\delta^{\frac{p}{p-1}}\Bigr]^{p-2} & \text{if } p\in(2,\infty),\end{cases}\tag{3.5}$$

which implies

$$\limsup_{\delta\downarrow 0}\sup_{g\in\mathcal{F};\mathcal{E}(g)\leq 1}\frac{D_{t,\delta}(f;g)}{\delta}\leq 0.\tag{3.6}$$

Since $\mathcal{E}$ is convex on $\mathcal{F}$, we know that the limits

$$\lim_{\delta\downarrow 0}\frac{\mathcal{E}\bigl(f+(t+\delta)g\bigr)-\mathcal{E}(f+tg)}{\delta}\quad\text{and}\quad\lim_{\delta\downarrow 0}\frac{\mathcal{E}\bigl(f+(t-\delta)g\bigr)-\mathcal{E}(f+tg)}{-\delta}$$

exist and

$$\frac{D_{t,\delta}(f;g)}{\delta} = \frac{\mathcal{E}\big(f+(t+\delta)g\big)+\mathcal{E}\big(f+(t-\delta)g\big)-2\mathcal{E}(f+tg)}{\delta} \geq 0. \tag{3.7}$$

By combining (3.6) and (3.7), we obtain (3.3) and the differentiability of $t \mapsto \mathcal{E}(f+tg)$. From the convexity of $t \mapsto \mathcal{E}(f+tg)$ again, we have

$$\sup_{g\in\mathcal{F};\mathcal{E}(g)\leq 1}\left|\frac{\mathcal{E}(f+(s+\delta)g)-\mathcal{E}(f+sg)}{\delta} - \frac{d}{dt}\mathcal{E}(f+tg)\Big|_{t=s}\right| \leq \sup_{g\in\mathcal{F};\mathcal{E}(g)\leq 1}\frac{D_{s,\delta}(f;g)}{\delta},$$

which together with (3.6) implies (3.2). □

Proposition 3.5, especially (3.2), implies the Fréchet differentiability of $\mathcal{E}$ on $\mathcal{F}/\mathcal{E}^{-1}(0)$. We record this fact and basic properties of these derivatives in the following theorem.

Theorem 3.6. *Assume that* $(\mathcal{E},\mathcal{F})$ *satisfies* (Cla)$_p$. *Then* $\mathcal{E}\colon \mathcal{F}/\mathcal{E}^{-1}(0) \to [0,\infty)$ *is Fréchet differentiable on the quotient normed space* $\mathcal{F}/\mathcal{E}^{-1}(0)$. *In particular, for any* $f,g \in \mathcal{F}$,

$$\textit{the derivative}\quad \mathcal{E}(f;g) := \frac{1}{p}\frac{d}{dt}\mathcal{E}(f+tg)\Big|_{t=0} \in \mathbb{R} \quad \textit{exists,} \tag{3.8}$$

the map $\mathcal{E}(f;\,\cdot\,)\colon \mathcal{F} \to \mathbb{R}$ *is linear,* $\mathcal{E}(f;f) = \mathcal{E}(f)$ *and* $\mathcal{E}(f;h) = 0$ *for* $h \in \mathcal{E}^{-1}(0)$. *Moreover, for any* $f, f_1, f_2, g \in \mathcal{F}$ *and any* $a \in \mathbb{R}$, *the following hold:*

$$\mathbb{R} \ni t \mapsto \mathcal{E}(f+tg;g) \in \mathbb{R} \textit{ is strictly increasing if and only if } g \notin \mathcal{E}^{-1}(0). \tag{3.9}$$

$$\mathcal{E}(af;g) = \operatorname{sgn}(a)\,|a|^{p-1}\,\mathcal{E}(f;g), \quad \mathcal{E}(f+h;g) = \mathcal{E}(f;g) \quad \textit{for } h \in \mathcal{E}^{-1}(0). \tag{3.10}$$

$$|\mathcal{E}(f;g)| \leq \mathcal{E}(f)^{(p-1)/p}\mathcal{E}(g)^{1/p}. \tag{3.11}$$

$$|\mathcal{E}(f_1;g)-\mathcal{E}(f_2;g)| \leq C_p\big(\mathcal{E}(f_1)\vee\mathcal{E}(f_2)\big)^{(p-1-\alpha_p)/p}\mathcal{E}(f_1-f_2)^{\alpha_p/p}\mathcal{E}(g)^{1/p}, \tag{3.12}$$

where $\alpha_p = \frac{1}{p} \wedge \frac{p-1}{p}$ *and some constant* $C_p \in (0,\infty)$ *determined solely and explicitly by* p.

Remark 3.7. It seems that the Hölder continuity exponent α_p appearing in (3.12) is not optimal because this exponent can be improved to $(p-1)\wedge 1$ in the case $\mathcal{E}(f;g) = \int_{\mathbb{R}^n} |\nabla f|^{p-2}\langle\nabla f,\nabla g\rangle\,dx$. However, such an improved continuity is unclear even for concrete p-energy forms constructed in the previous works [CGQ22, Kig23, MS23+, Shi24]. We can see the desired continuity ((3.12) with $(p-1)\wedge 1$ in place of α_p) for p-energy forms constructed in [KS.a], where a direct construction of p-energy forms based on the Korevaar–Schoen type p-energy forms is presented.

Proof. The existence of $\mathcal{E}(f;g)$ in (3.8) is already proved in Proposition 3.5. The properties $\mathcal{E}(f;ag) = a\mathcal{E}(f;g)$, $\mathcal{E}(af;g) = \operatorname{sgn}(a)\,|a|^{p-1}\,\mathcal{E}(f;g)$ and $\mathcal{E}(f;f) = \mathcal{E}(f)$ are obvious from the definition. The equalities $\mathcal{E}(f+h;g) = \mathcal{E}(f+g)$ and $\mathcal{E}(f;h) = 0$ for any $h \in \mathcal{E}^{-1}(0)$ follow from the triangle inequality for $\mathcal{E}^{1/p}$, so (3.10) holds. The property (3.9) is a consequence of the strict convexity of $\mathcal{E}$ (see (2.2)) and the differentiability in (3.8).

To show that $\mathcal{E}(f;\,\cdot\,)$ is linear, it suffices to prove $\mathcal{E}(f;g_1+g_2)=\mathcal{E}(f;g_1)+\mathcal{E}(f;g_2)$ for any $g_1,g_2\in\mathcal{F}$. For any $t>0$, the convexity of $\mathcal{E}$ implies that

$$\begin{aligned}\frac{\mathcal{E}\bigl(f+t(g_1+g_2)\bigr)-\mathcal{E}(f)}{t}&=\frac{\mathcal{E}\bigl(\frac{1}{2}(f+2tg_1)+\frac{1}{2}(f+2tg_2)\bigr)-\mathcal{E}(f)}{t}\\&\le\frac{\mathcal{E}(f+2tg_1)-\mathcal{E}(f)}{2t}+\frac{\mathcal{E}(f+2tg_2)-\mathcal{E}(f)}{2t}.\end{aligned}\tag{3.13}$$

Passing to the limit as $t\downarrow 0$, we get $\mathcal{E}(f;g_1+g_2)\le\mathcal{E}(f;g_1)+\mathcal{E}(f;g_2)$. We obtain the converse inequality by noting that

$$\frac{\mathcal{E}(f-tg)-\mathcal{E}(f)}{t}\to-\frac{d}{dt}\mathcal{E}(f+tg)\biggr|_{t=0}=-p\mathcal{E}(f;g)\quad\text{as } t\downarrow 0,$$

and by applying (3.13) with $-g_1,-g_2$ in place of g_1,g_2 respectively.

The Hölder-type estimate (3.11) follows from the following elementary estimate:

$$|a^q-b^q|=\left|\int_{a\wedge b}^{a\vee b}qt^{q-1}\,dt\right|\le q(a^{q-1}\vee b^{q-1})\,|a-b|\quad\text{for } q\in(0,\infty),\ a,b\in[0,\infty).\tag{3.14}$$

Indeed, by (3.14) and the triangle inequality for $\mathcal{E}^{1/p}$, for any $t>0$,

$$\left|\frac{\mathcal{E}(f+tg)-\mathcal{E}(f)}{t}\right|\le p\bigl(\mathcal{E}(f+tg)^{1/p}\vee\mathcal{E}(f)^{1/p}\bigr)^{p-1}\mathcal{E}(g)^{1/p}.\tag{3.15}$$

We obtain (3.11) by letting $t\downarrow 0$ in (3.15). We conclude that $\mathcal{E}(f;\,\cdot\,)$ is the Fréchet derivative of $\mathcal{E}$ at f by (3.2), the linearity of $\mathcal{E}(f;\,\cdot\,)$ and (3.11).

In the rest of this proof, we prove (3.12). Our proof is partially inspired by an argument by Šmulian in [Smu40]. In this proof, $C_{p,i}$, $i\in\{1,\dots,5\}$, is a constant depending only on p. We first show an analogue of (3.1) for $\mathcal{E}^{1/p}$. Using (3.14), we can show that there exists $c_*\in(0,2^{-p^3})$ depending only on p such that

$$\sup\left\{\frac{|\mathcal{E}(f)-\mathcal{E}(f+\delta g)|}{\mathcal{E}(f)}\;\middle|\;\begin{array}{l}f,g,\in\mathcal{F},\delta\in(0,\infty)\text{ such that}\\0<\delta<c_*\mathcal{E}(f)^{1/p}\text{ and }\mathcal{E}(g)=1\end{array}\right\}\le\frac{1}{10}.\tag{3.16}$$

Let $\psi(t)\coloneqq|t|^{1/p}$ and fix $g\in\mathcal{F}$ with $\mathcal{E}(g)=1$. Then there exist $\theta_1,\theta_2,\theta\in[0,1]$ such that

$$\begin{aligned}0&\le\psi(\mathcal{E}(f+\delta g))+\psi(\mathcal{E}(f-\delta g))-2\psi(\mathcal{E}(f))\\&=\psi'(A_{1,\delta})\bigl[\mathcal{E}(f+\delta g)-\mathcal{E}(f)\bigr]-\psi'(A_{2,\delta})\bigl[\mathcal{E}(f)-\mathcal{E}(f-\delta g)\bigr]\\&=\psi'(A_1(\delta))D_\delta(f;g)-\bigl(\psi'(A_{1,\delta})-\psi'(A_{2,\delta})\bigr)\bigl[\mathcal{E}(f)-\mathcal{E}(f-\delta g)\bigr]\\&=\psi'(A_{1,\delta})D_\delta(f;g)-\psi''\bigl(A_{1,\delta}+\theta(A_{2,\delta}-A_{1,\delta})\bigr)(A_{2,\delta}-A_{1,\delta})\bigl[\mathcal{E}(f)-\mathcal{E}(f-\delta g)\bigr],\end{aligned}\tag{3.17}$$

where $D_\delta(f;g)\coloneqq D_{\delta,0}(f;g)$ is the same as in (3.4) and

$$A_{1,\delta}\coloneqq\mathcal{E}(f)+\theta_1\bigl[\mathcal{E}(f+\delta g)-\mathcal{E}(f)\bigr],\quad A_{2,\delta}\coloneqq\mathcal{E}(f-\delta g)+\theta_2\bigl[\mathcal{E}(f)-\mathcal{E}(f-\delta g)\bigr].$$

By (3.16), we note that $|A_{1,\delta}| \wedge |A_{1,\delta} + \theta(A_{2,\delta} - A_{1,\delta})| \geq \frac{1}{2}\mathcal{E}(f)$, which together with (3.17) and (3.1) implies that for any $(\delta, f) \in (0,\infty) \times \mathcal{F}$ with $0 < \delta < c_{*}\mathcal{E}(f)^{1/p}$,

$$\begin{aligned} 0 &\leq \psi(\mathcal{E}(f+\delta g)) + \psi(\mathcal{E}(f-\delta g)) - 2\psi(\mathcal{E}(f)) \\ &\leq C_{p,1}\Bigl(\mathcal{E}(f)^{\frac{1}{p}-1+\frac{(p-2)^{+}}{p-1}}\delta^{p\wedge\frac{p}{p-1}} + \mathcal{E}(f)^{\frac{1}{p}-2+\frac{2(p-1)}{p}}\delta^{2}\Bigr) \\ &\leq C_{p,1}\delta \cdot \delta^{(p-1)\wedge\frac{1}{p-1}}\Bigl(\mathcal{E}(f)^{\frac{1}{p}-1+\frac{(p-2)^{+}}{p-1}} + \mathcal{E}(f)^{\frac{1}{p}-2+\frac{2(p-1)}{p}}\Bigr). \end{aligned}$$

In particular, if $\mathcal{E}(f) = 1$, then

$$\mathcal{E}(f+\delta g)^{1/p} + \mathcal{E}(f-\delta g)^{1/p} \leq 2 + C_{p,1}\delta^{(p-1)\wedge(p-1)^{-1}}\delta \quad \text{for any } \delta \in (0, c_{*}). \tag{3.18}$$

Next let $f_1, f_2 \in \mathcal{F}$. Then, by (3.11) and (3.14),

$$\begin{aligned} |\mathcal{E}(f_2; f_1) - \mathcal{E}(f_1)| &\leq |\mathcal{E}(f_2; f_1) - \mathcal{E}(f_2)| + |\mathcal{E}(f_2) - \mathcal{E}(f_1)| \\ &\leq \Bigl(\mathcal{E}(f_2)^{(p-1)/p} + p\Bigl(\mathcal{E}(f_2)^{(p-1)/p} \vee \mathcal{E}(f_1)^{(p-1)/p}\Bigr)\Bigr)\mathcal{E}(f_1 - f_2)^{1/p} \\ &\leq C_{p,2}\Bigl(\mathcal{E}(f_1)^{(p-1)/p} \vee \mathcal{E}(f_2)^{(p-1)/p}\Bigr)\mathcal{E}(f_1 - f_2)^{1/p}. \end{aligned} \tag{3.19}$$

Now, for any $f_1, f_2, g \in \mathcal{F}$ with $\mathcal{E}(f_1) = \mathcal{E}(g) = 1$ and $\delta \in (0, c_{*})$ we see that

$$\begin{aligned} &\mathcal{E}(f_1; \delta g) - \mathcal{E}(f_2; \delta g) \\ &= \mathcal{E}(f_1; f_1 + \delta g) + \mathcal{E}(f_2; f_1 - \delta g) - \mathcal{E}(f_1) - \mathcal{E}(f_2; f_1) \\ &\overset{(3.11)}{\leq} \Bigl(\mathcal{E}(f_1)^{(p-1)/p} \vee \mathcal{E}(f_2)^{(p-1)/p}\Bigr)\Bigl(\mathcal{E}(f_1+\delta g)^{1/p} + \mathcal{E}(f_1 - \delta g)^{1/p}\Bigr) - \mathcal{E}(f_1) - \mathcal{E}(f_2; f_1) \\ &\overset{(3.14),(3.18)}{\leq} \Bigl(1 + C_{p,3}\mathcal{E}(f_1 - f_2)^{1/p}\Bigr)\Bigl(2 + C_{p,1}\delta^{(p-1)\wedge(p-1)^{-1}}\delta\Bigr) - \mathcal{E}(f_1) - \mathcal{E}(f_2; f_1). \end{aligned}$$

Similarly, we can show

$$\begin{aligned} &\mathcal{E}(f_1; \delta g) - \mathcal{E}(f_2; \delta g) \\ &= -\mathcal{E}(f_1; f_1 - \delta g) - \mathcal{E}(f_2; f_1 + \delta g) + \mathcal{E}(f_1) + \mathcal{E}(f_2; f_1) \\ &\geq -\Bigl(1 + C_{p,3}\mathcal{E}(f_1 - f_2)^{1/p}\Bigr)\Bigl(2 + C_{p,1}\delta^{(p-1)\wedge(p-1)^{-1}}\delta\Bigr) + \mathcal{E}(f_1) + \mathcal{E}(f_2; f_1). \end{aligned}$$

From these estimates, we have

$$\begin{aligned} &|\mathcal{E}(f_1; g) - \mathcal{E}(f_2; g)| = \frac{|\mathcal{E}(f_1; \delta g) - \mathcal{E}(f_2; \delta g)|}{\delta} \\ &\leq \Bigl(1 + C_{p,3}\mathcal{E}(f_1 - f_2)^{1/p}\Bigr)\Bigl(2\delta^{-1} + C_{p,1}\delta^{(p-1)\wedge(p-1)^{-1}}\Bigr) - \delta^{-1}\mathcal{E}(f_1) - \delta^{-1}\mathcal{E}(f_2; f_1) \\ &= \Bigl(1 + C_{p,3}\mathcal{E}(f_1 - f_2)^{1/p}\Bigr)\Bigl(2\delta^{-1} + C_{p,1}\delta^{(p-1)\wedge(p-1)^{-1}}\Bigr) - 2\delta^{-1}\mathcal{E}(f_1) + \delta^{-1}\bigl(\mathcal{E}(f_1) - \mathcal{E}(f_2; f_1)\bigr) \\ &\overset{(3.19)}{\leq} \Bigl(1 + C_{p,3}\mathcal{E}(f_1 - f_2)^{1/p}\Bigr)\Bigl(2\delta^{-1} + C_{p,1}\delta^{(p-1)\wedge(p-1)^{-1}}\Bigr) - 2\delta^{-1} + C_{p,2}\delta^{-1}\mathcal{E}(f_1 - f_2)^{1/p} \\ &\leq C_{p,4}\Bigl(\delta^{(p-1)\wedge(p-1)^{-1}} + \delta^{-1}\mathcal{E}(f_1 - f_2)^{1/p}\Bigr). \end{aligned}$$

If $\mathcal{E}(f_1 - f_2) < c_*^{-p^2/((p-1)\vee 1)}$, then, by choosing $\delta = \mathcal{E}(f_1 - f_2)^{((p-1)\vee 1)/p^2}$, we obtain

$$|\mathcal{E}(f_1; g) - \mathcal{E}(f_2; g)| \leq C_{p,5}\mathcal{E}(f_1 - f_2)^{((p-1)\wedge 1)/p^2}. \tag{3.20}$$

The same is clearly true if $\mathcal{E}(f_1 - f_2) \geq c_*^{-p^2/((p-1)\vee 1)}$ since $\mathcal{E}(f_2) \leq 2^{p-1}\bigl(1 + \mathcal{E}(f_1 - f_2)\bigr)$. Finally, for any $f_1, f_2, g \in \mathcal{F}$ with $\mathcal{E}(f_1) \wedge \mathcal{E}(g) > 0$, we have

$$\begin{aligned}
|\mathcal{E}(f_1; g) - \mathcal{E}(f_2; g)| &= \mathcal{E}(f_1)^{(p-1)/p}\mathcal{E}(g)^{1/p}\left|\mathcal{E}\left(\frac{f_1}{\mathcal{E}(f_1)^{1/p}}; \frac{g}{\mathcal{E}(g)^{1/p}}\right) - \mathcal{E}\left(\frac{f_2}{\mathcal{E}(f_1)^{1/p}}; \frac{g}{\mathcal{E}(g)^{1/p}}\right)\right| \\
&\overset{(3.20)}{\leq} C_{p,5}\mathcal{E}(f_1)^{(p-1)/p}\mathcal{E}(g)^{1/p}\mathcal{E}\left(\frac{f_1}{\mathcal{E}(f_1)^{1/p}} - \frac{f_2}{\mathcal{E}(f_1)^{1/p}}\right)^{((p-1)\wedge 1)/p^2} \\
&\overset{(3.20)}{\leq} C_{p,5}\bigl(\mathcal{E}(f_1) \vee \mathcal{E}(f_2)\bigr)^{(p-1-\alpha_p)/p}\mathcal{E}(g)^{1/p}\mathcal{E}(f_1 - f_2)^{\alpha_p/p}.
\end{aligned}$$

The same estimate is clearly true if $\mathcal{E}(f_2) \wedge \mathcal{E}(g) > 0$. Since (3.12) is obvious when $g \in \mathcal{E}^{-1}(0)$ or $\mathcal{E}(f_1) \vee \mathcal{E}(f_2) = 0$, we obtain (3.12). □

The following theorem gives a quantitative continuity for the inverse map of $f \mapsto \mathcal{E}(f; \cdot)$.

Theorem 3.8. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies* (Cla)$_p$. *Then for any $f, g \in \mathcal{F}$,*

$$\mathcal{E}(f-g) \leq C'_p\Bigl[\bigl(\mathcal{E}(f)\vee\mathcal{E}(g)\bigr)^{1/p}\vee\bigl(\mathcal{E}(f)\vee\mathcal{E}(g)\bigr)^{\alpha'_p}\Bigr]\left(\sup_{\varphi\in\mathcal{F};\mathcal{E}(\varphi)\leq 1}|\mathcal{E}(f;\varphi) - \mathcal{E}(g;\varphi)|\right), \tag{3.21}$$

where $\alpha'_p = \frac{1}{p} + \frac{p(p-2)^+}{p-1}$ and some constant $C'_p \in (0, \infty)$ determined solely and explicitly by p.

Proof. For simplicity, for any linear functional $\Phi\colon \mathcal{F} \to \mathbb{R}$, we set $\|\Phi\|_{\mathcal{F},*} := \sup_{u\in\mathcal{F};\mathcal{E}(u)\leq 1}|\Phi(u)|$. Clearly, $\|\Phi_1 + \Phi_2\|_{\mathcal{F},*} \leq \|\Phi_1\|_{\mathcal{F},*} + \|\Phi_2\|_{\mathcal{F},*}$ for any linear functionals $\Phi_1, \Phi_2\colon \mathcal{F} \to \mathbb{R}$. Note that $\|\mathcal{E}(f; \cdot)\|_{\mathcal{F},*} = \mathcal{E}(f)^{(p-1)/p}$ by (3.11) for any $f \in \mathcal{F}$. In particular, for any $f, g \in \mathcal{F}$,

$$\left|\mathcal{E}(f)^{\frac{p-1}{p}} - \mathcal{E}(g)^{\frac{p-1}{p}}\right| = \left|\|\mathcal{E}(f; \cdot)\|_{\mathcal{F},*} - \|\mathcal{E}(g; \cdot)\|_{\mathcal{F},*}\right| \leq \|\mathcal{E}(f; \cdot) - \mathcal{E}(g; \cdot)\|_{\mathcal{F},*},$$

which together with (3.14) with $q = (p-1)/p$ implies that

$$|\mathcal{E}(f) - \mathcal{E}(g)| \leq \frac{p}{p-1}\bigl(\mathcal{E}(f)^{1/p} \vee \mathcal{E}(g)^{1/p}\bigr)\,\|\mathcal{E}(f; \cdot) - \mathcal{E}(g; \cdot)\|_{\mathcal{F},*}. \tag{3.22}$$

Let us define $\psi\colon \mathbb{R} \to \mathbb{R}$ by $\psi(t) := p^{-1}\mathcal{E}(f + t(g - f))$. Then $\psi \in C^1(\mathbb{R})$ by (3.2) and (3.12); indeed, (3.2) implies that $\psi'(t) = \mathcal{E}(f + t(g - f); g - f)$, which is continuous by (3.12). Now we see that

$$|\psi'(0)| = |\mathcal{E}(f; g - f)| \leq |\mathcal{E}(f; g) - \mathcal{E}(g)| + |\mathcal{E}(g) - \mathcal{E}(f)|$$

$$\overset{(3.22)}{\leq} \|\mathcal{E}(f;\,\cdot\,) - \mathcal{E}(g;\,\cdot\,)\|_{\mathcal{F},*}\,\mathcal{E}(g)^{1/p} + \frac{p}{p-1}\big(\mathcal{E}(f)^{1/p} \vee \mathcal{E}(g)^{1/p}\big)\,\|\mathcal{E}(f;\,\cdot\,) - \mathcal{E}(g;\,\cdot\,)\|_{\mathcal{F},*}$$
$$\leq \left(1 + \frac{p}{p-1}\right)\big(\mathcal{E}(f)^{1/p} \vee \mathcal{E}(g)^{1/p}\big)\,\|\mathcal{E}(f;\,\cdot\,) - \mathcal{E}(g;\,\cdot\,)\|_{\mathcal{F},*}.$$

Similarly,

$$|\psi'(1)| = |\mathcal{E}(g; g - f)| \leq \left(1 + \frac{p}{p-1}\right)\big(\mathcal{E}(f)^{1/p} \vee \mathcal{E}(g)^{1/p}\big)\,\|\mathcal{E}(f;\,\cdot\,) - \mathcal{E}(g;\,\cdot\,)\|_{\mathcal{F},*}.$$

Since ψ is C^1-convex, we obtain

$$\max_{t\in[0,1]} |\psi'(t)| \leq \left(1 + \frac{p}{p-1}\right)\big(\mathcal{E}(f)^{1/p} \vee \mathcal{E}(g)^{1/p}\big)\,\|\mathcal{E}(f;\,\cdot\,) - \mathcal{E}(g;\,\cdot\,)\|_{\mathcal{F},*},$$

and hence

$$\left|-\mathcal{E}(f) + \mathcal{E}\left(\frac{f+g}{2}\right)\right| = |-\mathcal{E}(f) + p\psi(1/2)| = p\,|\psi(1/2) - \psi(0)| \leq \frac{p}{2}\,|\psi'(1)|$$
$$\leq c_p\big(\mathcal{E}(f)^{1/p} \vee \mathcal{E}(g)^{1/p}\big)\,\|\mathcal{E}(f;\,\cdot\,) - \mathcal{E}(g;\,\cdot\,)\|_{\mathcal{F},*},$$

where we put $c_p \coloneqq \frac{p}{2}\big(1 + \frac{p}{p-1}\big)$. Similarly,

$$\left|-\mathcal{E}(g) + \mathcal{E}\left(\frac{f+g}{2}\right)\right| \leq \frac{p}{2}\,|\psi'(0)| \leq c_p\big(\mathcal{E}(f)^{1/p} \vee \mathcal{E}(g)^{1/p}\big)\,\|\mathcal{E}(f;\,\cdot\,) - \mathcal{E}(g;\,\cdot\,)\|_{\mathcal{F},*}.$$

Therefore, it follows that

$$\mathcal{E}\left(\frac{f+g}{2}\right) \geq \Big(\mathcal{E}(f) \vee \mathcal{E}(g) - c_p\big(\mathcal{E}(f)^{1/p} \vee \mathcal{E}(g)^{1/p}\big)\,\|\mathcal{E}(f;\,\cdot\,) - \mathcal{E}(g;\,\cdot\,)\|_{\mathcal{F},*}\Big)^{+}. \tag{3.23}$$

Next we derive an estimate on $\mathcal{E}(\frac{f-g}{2})$ by using (Cla)$_p$ and (3.23). Set $a \coloneqq \mathcal{E}(f) \vee \mathcal{E}(g)$ for simplicity. If $p \in [2,\infty)$, then

$$\mathcal{E}\left(\frac{f-g}{2}\right) \overset{(\mathrm{Cla})_p}{\leq} 2^{1-p}\big(\mathcal{E}(f)^{1/(p-1)} + \mathcal{E}(g)^{1/(p-1)}\big)^{p-1} - \mathcal{E}\left(\frac{f+g}{2}\right)$$
$$\overset{(3.23)}{\leq} a - \Big(a - c_p a^{1/p}\,\|\mathcal{E}(f;\,\cdot\,) - \mathcal{E}(g;\,\cdot\,)\|_{\mathcal{F},*}\Big)^{+}$$
$$\leq c_p a^{1/p}\,\|\mathcal{E}(f;\,\cdot\,) - \mathcal{E}(g;\,\cdot\,)\|_{\mathcal{F},*}.$$

In the rest of the proof, we assume that $p \in (1,2]$. We see that

$$\mathcal{E}\left(\frac{f-g}{2}\right)^{1/(p-1)} \overset{(\mathrm{Cla})_p}{\leq} \left(\frac{\mathcal{E}(f) + \mathcal{E}(g)}{2}\right)^{1/(p-1)} - \mathcal{E}\left(\frac{f+g}{2}\right)^{1/(p-1)}$$
$$\overset{(3.23)}{\leq} a^{1/(p-1)} - \left[\Big(a - c_p a^{1/p}\,\|\mathcal{E}(f;\,\cdot\,) - \mathcal{E}(g;\,\cdot\,)\|_{\mathcal{F},*}\Big)^{+}\right]^{1/(p-1)}. \tag{3.24}$$

In the case $a \leq c_p a^{1/p} \left\| \mathcal{E}(f;\,\cdot\,) - \mathcal{E}(g;\,\cdot\,) \right\|_{\mathcal{F},*}$, we have

$$\mathcal{E}\left(\frac{f-g}{2}\right) \leq a \leq c_p a^{1/p} \left\| \mathcal{E}(f;\,\cdot\,) - \mathcal{E}(g;\,\cdot\,) \right\|_{\mathcal{F},*}.$$

Let us consider the remaining case $a > c_p a^{1/p} \left\| \mathcal{E}(f;\,\cdot\,) - \mathcal{E}(g;\,\cdot\,) \right\|_{\mathcal{F},*}$. Then we have from (3.14) with $q = \frac{1}{p-1}$ that

$$\begin{aligned} \mathcal{E}\left(\frac{f-g}{2}\right)^{1/(p-1)} &= a^{1/(p-1)} - \left(a - c_p a^{1/p} \left\| \mathcal{E}(f;\,\cdot\,) - \mathcal{E}(g;\,\cdot\,) \right\|_{\mathcal{F},*}\right)^{1/(p-1)} \\ &\leq \frac{c_p}{p-1} a^{\frac{2-p}{p-1}+\frac{1}{p}} \left\| \mathcal{E}(f;\,\cdot\,) - \mathcal{E}(g;\,\cdot\,) \right\|_{\mathcal{F},*}. \end{aligned}$$

Hence we obtain the desired estimate (3.21). □

The following proposition is a kind of 'monotonicity on values of p-Laplacian'. This result will play important roles in Subsection 6.4 later and in the subsequent works [KS.b, KS.c].

Proposition 3.9. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies* (Cla)$_p$ *and the strong subadditivity* (2.5). *Let $u_1, u_2, v \in \mathcal{F}$ satisfy $((u_2 - u_1) \wedge v)(x) = 0$ for m-a.e. $x \in X$. Then $\mathcal{E}(u_1; v) \geq \mathcal{E}(u_2; v)$.*

Proof. Let $t > 0$. Define $f, g \in \mathcal{F}$ by $f := u_1 + tv$ and $g := u_2$. Then we easily see that $f \vee g = u_2 + tv$ and $f \wedge g = u_1$. By (2.5), we have $\mathcal{E}(u_2 + tv) + \mathcal{E}(u_1) \leq \mathcal{E}(u_1 + tv) + \mathcal{E}(u_2)$, which implies that

$$\frac{\mathcal{E}(u_2 + tv) - \mathcal{E}(u_2)}{t} \leq \frac{\mathcal{E}(u_1 + tv) - \mathcal{E}(u_1)}{t}.$$

Letting $t \downarrow 0$, we get $\mathcal{E}(u_2; v) \leq \mathcal{E}(u_1; v)$. □

We conclude this subsection by viewing typical examples of p-energy forms.

Example 3.10. (1) Let $D \in \mathbb{N}$, let $X := \Omega \subseteq \mathbb{R}^D$ be a domain, let $\mathcal{B} := \mathcal{B}(X)$, let m be the D-dimensional Lebesgue measure on X and let $\mathcal{F} = W^{1,p}(\Omega)$ be the usual $(1,p)$-Sobolev space on Ω (see [AF, p. 60] for example). Define $\mathcal{E}(f) := \|\nabla f\|_{L^p(X,m)}^p$, $f \in \mathcal{F}$, where the gradient operator ∇ is regarded in the distribution sense. Then , by following a similar argument as in the proof of Theorem A.19, one can show that $(\mathcal{E}, \mathcal{F})$ is a p-energy form on (X, m) satisfying (GC)$_p$. In this case, we have

$$\mathcal{E}(f; g) = \int_\Omega |\nabla f(x)|^{p-2} \langle \nabla f(x), \nabla g(x) \rangle_{\mathbb{R}^D}\, dx, \quad f, g \in \mathcal{F},$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{R}^D}$ denotes the inner product on $\mathbb{R}^D$.

(2) In the recent work [Kig23, MS23+], a p-energy form $(\mathcal{E}, \mathcal{F})$ on a compact metrizable space with some geometric assumptions is constructed via discrete approximations. See [HPS04, CGQ22] for constructions of p-energy forms on post-critically finite self-similar sets. The construction in [CGQ22] can be seen as a generalization of that in

[HPS04]. As will be seen in more detail later in Section 8, we can prove that p-energy forms constructed in [CGQ22, Kig23, MS23+] satisfy $(\mathrm{GC})_p$ although even $(\mathrm{Cla})_p$ is not mentioned in [CGQ22, Kig23]. Furthermore, very recently, Kuwae [Kuw24] introduced a p-energy form $(\mathscr{E}^p, H^{1,p})$ based on a strongly local Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ on $L^2(X,m)$. It is shown that $(\mathscr{E}^p, H^{1,p})$ satisfies $(\mathrm{Cla})_p$ in [Kuw24, Theorem 1.8]. We can also verify $(\mathrm{GC})_p$ for $(\mathscr{E}^p, H^{1,p})$ (see Theorem A.19) by using some good estimates due to the bilinearity. See Appendix A for details.

(3) There are various ways to define $(1,p)$-Sobolev spaces in the field of 'Analysis on metric spaces' (see, e.g., [HKST, Chapter 10]). In these definitions, roughly speaking, we find a counterpart of $|\nabla u|$, e.g., the minimal p-weak upper gradient $g_u \geq 0$ (see, e.g., [HKST, Chapter 6] for details), and consider a p-energy form $(\widetilde{\mathcal{E}}, \mathcal{F})$ on (X,m) given by $\widetilde{\mathcal{E}}(u) \coloneqq \int_X g_u^p \, dm$ and $\mathcal{F} \coloneqq \{u \in L^p(X,m) \mid g_u \in L^p(X,m)\}$. Unfortunately, this p-energy form may not satisfy $(\mathrm{Cla})_p$ because of a lack of the linearity of $u \mapsto g_u$ (see, e.g., [HKST, (6.3.18)]). However, in a suitable setting, we can construct a functional which is equivalent to $\widetilde{\mathcal{E}}$ and satisfies $(\mathrm{Cla})_p$; see the p-energy form denoted by $(\mathcal{F}_p, W^{1,p})$ in [ACD15, Theorem 40]. Moreover, we can verify $(\mathrm{GC})_p$ for $(\mathcal{F}_p, W^{1,p})$ since $(\mathcal{F}_{\delta_k,p}, W^{1,p})$ defined in [ACD15, (7.3)] satisfies $(\mathrm{GC})_p$ and $\mathcal{F}_p$ is defined as a Γ-limit point of $\mathcal{F}_{\delta_k,p}$ as $k \to \infty$. (See also the proof of Theorem 8.19 later.)

3.2 p-Clarkson's inequality and approximations in p-energy forms

In this subsection, in addition to the setting specified at the beginning of this section, by considering $\mathcal{F} \cap L^p(X,m)$ instead of $\mathcal{F}$ if necessary, we also assume that $\mathcal{F} \subseteq L^p(X,m)$ for simplicity.

We introduce a family of natural norms on $\mathcal{F}$ in the following definition.

Definition 3.11 ($(\mathcal{E},\alpha)$-norm)**.** Let $\alpha \in (0,\infty)$. We define the norm $\|\cdot\|_{\mathcal{E},\alpha}$ on $\mathcal{F}$ by

$$\|f\|_{\mathcal{E},\alpha} \coloneqq \Bigl(\mathcal{E}(f) + \alpha \|f\|_{L^p(X,m)}^p\Bigr)^{1/p}, \quad f \in \mathcal{F} \tag{3.25}$$

We call $\|\cdot\|_{\mathcal{E},\alpha}$ the $(\mathcal{E},\alpha)$-*norm* on $\mathcal{F}$.

The following proposition states on the convexity of $\|\cdot\|_{\mathcal{E},\alpha}$.

Proposition 3.12. *Let $\alpha \in (0,\infty)$ and assume that $(\mathcal{E},\mathcal{F})$ satisfies $(\mathrm{Cla})_p$. Then $(\|\cdot\|_{\mathcal{E},\alpha}^p, \mathcal{F})$ is a p-energy form on (X,m) satisfying $(\mathrm{Cla})_p$, and $(\mathcal{F}, \|\cdot\|_{\mathcal{E},\alpha})$ is uniformly convex. If $(\mathcal{F}, \|\cdot\|_{\mathcal{E},\alpha})$ is a Banach space in addition, then it is reflexive.*

Proof. We have $(\mathrm{Cla})_p$ for the p-energy form $(\|\cdot\|_{\mathcal{E},\alpha}^p, \mathcal{F})$ on (X,m) by applying (2.20) for $T\colon \mathbb{R}^2 \to \mathbb{R}$ given in Proposition 2.2-(e),(f). The uniform convexity $\|\cdot\|_{\mathcal{E},\alpha}$ follows from [Cla36, Proof of Corollary of Theorem 2].

Assume that $(\mathcal{F}, \|\cdot\|_{\mathcal{E},\alpha})$ is a Banach space. Then $(\mathcal{F}, \|\cdot\|_{\mathcal{E},\alpha})$ is reflexive by the Milman–Pettis theorem (see, e.g., [LT, Proposition 1.e.3]) since $(\mathcal{F}, \|\cdot\|_{\mathcal{E},\alpha})$ is uniformly convex. □

We will frequently use the following Mazur's lemma, which is an elementary fact in the theory of Banach spaces.

Lemma 3.13 (Mazur's lemma; see, e.g., [HKST, p. 19])**.** *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in a normed space V converging weakly to some element $v \in V$. Then there exist $\{N_l\}_{l\in\mathbb{N}} \subseteq \mathbb{N}$ with $N_l > l$ and $\{\lambda_{k,l} \in [0,1] \mid k = l, l+1, \dots, N_l\}$ with $\sum_{k=l}^{N_l} \lambda_{k,l} = 1$ such that $\sum_{k=l}^{N_l} \lambda_{k,l} v_k$ converges strongly to v as $l \to \infty$.*

We also prepare the following two lemmas.

Lemma 3.14. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{Cla})_p$ and that $\mathcal{F}$ equipped with $\|\cdot\|_{\mathcal{E},1}$ is a Banach space. For $v \in L^{\frac{p}{p-1}}(X,m)$, we define a bounded linear map $\Psi_v \colon L^p(X,m) \to \mathbb{R}$ by $\Psi_v(u) \coloneqq \int_X uv\,dm$. Then $\{\Psi_v|_{\mathcal{F}} \mid v \in L^{\frac{p}{p-1}}(X,m)\}$ is dense in $\mathcal{F}^*$.*

Proof. Set $M \coloneqq \{\Psi_v|_{\mathcal{F}} \mid v \in L^{\frac{p}{p-1}}(X,m)\}$ for simplicity. Then $M \subseteq \mathcal{F}^*$ since $\|u\|_{L^p(X,m)} \le \|u\|_{\mathcal{E},1}$ for any $u \in \mathcal{F}$. Suppose that $\overline{M}^{\mathcal{F}^*} \neq \mathcal{F}^*$. Let $\varphi \in \mathcal{F}^* \setminus \overline{M}^{\mathcal{F}^*}$. By the Hahn–Banach theorem, there exists $\Phi \in \mathcal{F}^{**}$ such that $\Phi(\varphi) \neq 0$ and $\Phi|_{\overline{M}^{\mathcal{F}^*}} = 0$. Since $\mathcal{F}$ is reflexive by Proposition 3.12, there exists $u \in \mathcal{F}$ such that $\Phi(\psi) = \Psi(u)$ for any $\psi \in \mathcal{F}^*$. Then for any $\psi \in M$, we have $\psi(u) = \Phi(\psi) = 0$, which implies that $u = 0$. This contradicts $\varphi(u) = \Phi(\varphi) \neq 0$. □

Lemma 3.15. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{Cla})_p$ and that $\mathcal{F}$ equipped with $\|\cdot\|_{\mathcal{E},1}$ is a Banach space. If $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ converges in $L^p(X,m)$ to $u \in \mathcal{F}$ and $\sup_{n\in\mathbb{N}} \mathcal{E}(u_n) < \infty$, then $\{u_n\}_{n\in\mathbb{N}}$ converges weakly in $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ to u.*

Proof. For any $\varphi \in \mathcal{F}^*$ and any $\varepsilon > 0$, by Lemma 3.14, there exists $v \in L^{\frac{p}{p-1}}(X,m)$ such that $\|\varphi - \Psi_v|_{\mathcal{F}}\|_{\mathcal{F}^*} < \varepsilon$. Then we easily see that

$$\begin{aligned}
|\varphi(u) - \varphi(u_n)| &\le |\varphi(u) - \Psi_v(u)| + |\Psi_v(u) - \Psi_v(u_n)| + |\varphi(u_n) - \Psi_v(u_n)| \\
&\le \varepsilon\left(\|u\|_{\mathcal{E},1} + \sup_{n\in\mathbb{N}} \|u_n\|_{\mathcal{E},1}\right) + |\Psi_v(u) - \Psi_v(u_n)|,
\end{aligned}$$

whence $\limsup_{n\to\infty} |\varphi(u) - \varphi(u_n)| \le \varepsilon\bigl(\|u\|_{\mathcal{E},1} + \sup_{n\in\mathbb{N}} \|u_n\|_{\mathcal{E},1}\bigr)$. Since $\varepsilon > 0$ is arbitrary, we obtain $\lim_{n\to\infty} \varphi(u_n) = \varphi(u)$. This completes the proof. □

We collect some useful results on converges in $\mathcal{E}$ in the following proposition. Let us regard $\mathcal{E}$ as a $[0,\infty]$-valued functional on $L^p(X,m)$ by setting $\mathcal{E}(f) \coloneqq \infty$ for $f \in L^p(X,m) \setminus \mathcal{F}$.

Proposition 3.16. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{Cla})_p$ and that $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ is a Banach space.*

(a) *If $\{u_n\}_{n\in\mathbb{N}} \subseteq L^p(X,m)$ converges to $u \in L^p(X,m)$ in $L^p(X,m)$ as $n \to \infty$, then $\mathcal{E}(u) \le \liminf_{n\to\infty} \mathcal{E}(u_n)$.*

(b) *If $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ converges to $u \in \mathcal{F}$ in $L^p(X,m)$ as $n \to \infty$ and $\lim_{n\to\infty} \mathcal{E}(u_n) = \mathcal{E}(u)$, then $u \in \mathcal{F}$ and $\lim_{n\to\infty} \|u - u_n\|_{\mathcal{E},1} = 0$.*

Proof. (a): If $\liminf_{n\to\infty}\mathcal{E}(u_n)=\infty$, then the desired statement clearly holds. So, we assume that $\liminf_{n\to\infty}\mathcal{E}(u_n)<\infty$. Pick a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\mathcal{E}(u_{n_k})=\liminf_{n\to\infty}\mathcal{E}(u_n)$. Then $\{u_{n_k}\}_k$ is a bounded sequence in $(\mathcal{F},\|\cdot\|_{\mathcal{E},1})$ and hence Lemma 3.15 implies that $\{u_{n_k}\}_{k\in\mathbb{N}}$ converges weakly in $\mathcal{F}$ to u. Since $\|\cdot\|_{\mathcal{E},1}$ is lower semicontinuous with respect to the weak topology of $\mathcal{F}$, we have from $\lim_{k\to\infty}\|u_{n_k}\|_{L^p(X,m)}=\|u\|_{L^p(X,m)}$ that $\mathcal{E}(u)^{1/p}\le\liminf_{n\to\infty}\mathcal{E}(u_n)^{1/p}$.

(b): If $u\in\mathcal{E}^{-1}(0)$, then $\mathcal{E}(u-u_n)=\mathcal{E}(u_n)\to\mathcal{E}(u)=0$. It suffices to consider the case $\mathcal{E}(u)=1$. Since $u+u_n$ converges in $L^p(X,m)$ to $2u$ as $n\to\infty$, by (a),

$$\begin{aligned}2=\mathcal{E}(2u)^{1/p}\le\liminf_{n\to\infty}\mathcal{E}\bigl(u+u_n\bigr)^{1/p}&\le\limsup_{n\to\infty}\mathcal{E}\bigl(u+u_n\bigr)^{1/p}\\&\le\lim_{n\to\infty}\mathcal{E}(u_n)^{1/p}+\mathcal{E}(u)^{1/p}=2,\end{aligned}$$

i.e., $\lim_{n\to\infty}\mathcal{E}(u+u_n)=2^p$. By $(\mathrm{Cla})_p$, if $p\le 2$, then

$$\begin{aligned}\lim_{n\to\infty}\mathcal{E}(u-u_n)^{1/(p-1)}&\le 2\Bigl(\mathcal{E}(u)+\lim_{n\to\infty}\mathcal{E}(u_n)\Bigr)^{1/(p-1)}-\lim_{n\to\infty}\mathcal{E}(u+u_n)^{1/(p-1)}\\&=2\cdot 2^{1/(p-1)}-2^{p/(p-1)}=0.\end{aligned}$$

If $p\ge 2$, then

$$\lim_{n\to\infty}\mathcal{E}(u-u_n)\le 2^{p-1}\Bigl(\mathcal{E}(u)+\lim_{n\to\infty}\mathcal{E}(u_n)\Bigr)-\lim_{n\to\infty}\mathcal{E}(u+u_n)=2^{p-1}\cdot 2-2^p=0.$$

Since u_n converges in $L^p(X,m)$ to u as $n\to\infty$, we obtain the desired convergence. □

The following convergences in $\mathcal{E}$ are also useful. These are analogues of [FOT, Theorem 1.4.2-(iii), Theorem 1.4.2-(v)].

Corollary 3.17. *Assume that $(\mathcal{E},\mathcal{F})$ satisfies $(\mathrm{Cla})_p$ and that $(\mathcal{F},\|\cdot\|_{\mathcal{E},1})$ is a Banach space. In addition, we assume the following property: if $\varphi\in C(\mathbb{R})$ satisfies $\varphi(0)=0$ and $|\varphi(t)-\varphi(s)|\le|t-s|$ for any $s,t\in\mathbb{R}$, then $\varphi(u)\in\mathcal{F}$ and $\mathcal{E}(\varphi(u))\le\mathcal{E}(u)$ for any $u\in\mathcal{F}$.*

(a) *Let $\{\varphi_n\}_{n\in\mathbb{N}}\subseteq C(\mathbb{R})$ satisfy $\lim_{n\to\infty}\varphi_n(t)=t$, $\varphi_n(0)=0$ and $|\varphi_n(t)-\varphi_n(s)|\le|t-s|$ for any $n\in\mathbb{N}$, $s,t\in\mathbb{R}$. Then $\{\varphi_n(u)\}_{n\in\mathbb{N}}\subseteq\mathcal{F}$ and $\lim_{n\to\infty}\mathcal{E}(u-\varphi_n(u))=0$ for any $u\in\mathcal{F}$.*

(b) *Let $u\in\mathcal{F}$, $\{u_n\}_{n\in\mathbb{N}}\subseteq\mathcal{F}$ and $\varphi\in C(\mathbb{R})$ satisfy $\lim_{n\to\infty}\|u-u_n\|_{\mathcal{E},1}=0$, $\varphi(0)=0$, $|\varphi(t)-\varphi(s)|\le|t-s|$ for any $s,t\in\mathbb{R}$ and $\varphi(u)=u$. Then $\{\varphi(u_n)\}_{n\in\mathbb{N}}\subseteq\mathcal{F}$ and $\lim_{n\to\infty}\mathcal{E}(u-\varphi(u_n))=0$.*

Remark 3.18. Let us make the same remark as [KS23+, Remark 2.21] for convenience. Typical choices of $\{\varphi_n\}_{n\in\mathbb{N}}\subseteq C(\mathbb{R})$ in Corollary 3.17-(a) are $\varphi_n(t)=(-n)\vee(t\wedge n)$ and $\varphi_n(t)=t-(-\frac{1}{n})\vee(t\wedge\frac{1}{n})$. A typical use of Corollary 3.17-(b) is to obtain a sequence of I-valued functions converging to u in $(\mathcal{F},\|\cdot\|_{\mathcal{E},1})$ when $I\subseteq\mathbb{R}$ is a closed interval and $u\in\mathcal{F}$ is I-valued, by considering $\varphi\in C(\mathbb{R})$ given by $\varphi(t):=(\inf I)\vee(t\wedge\sup I)$.

Proof. (a): We have $\varphi_n(u) \in \mathcal{F}$ by the assumption on $(\mathcal{E}, \mathcal{F})$. It is immediate from the dominated convergence theorem that $\varphi_n(u)$ converges in $L^p(X, m)$ to u as $n \to \infty$. By $\mathcal{E}(\varphi_n(u)) \le \mathcal{E}(u)$ and Proposition 3.16-(a),

$$\mathcal{E}(u) \le \liminf_{n\to\infty} \mathcal{E}(u_n) \le \limsup_{n\to\infty} \mathcal{E}(u_n) \le \mathcal{E}(u),$$

which implies $\lim_{n\to\infty} \mathcal{E}(u_n) = \mathcal{E}(u)$. We have $\lim_{n\to\infty} \mathcal{E}(u - \varphi_n(u)) = 0$ by Proposition 3.16-(b).

(b): By the dominated convergence theorem, $\varphi(u_n)$ converges in $L^p(X, m)$ to $\varphi(u) = u$ as $n \to \infty$. We have $\varphi(u_n) \in \mathcal{F}$ by the assumption on $(\mathcal{E}, \mathcal{F})$. By $\mathcal{E}(\varphi(u_n)) \le \mathcal{E}(u_n)$ and Proposition 3.16-(a),

$$\mathcal{E}(u) = \mathcal{E}(\varphi(u)) \le \liminf_{n\to\infty} \mathcal{E}(\varphi(u_n)) \le \limsup_{n\to\infty} \mathcal{E}(\varphi(u_n)) \le \lim_{n\to\infty} \mathcal{E}(u_n) = \mathcal{E}(u),$$

which implies $\lim_{n\to\infty} \mathcal{E}(\varphi(u_n)) = \mathcal{E}(u)$. We have $\lim_{n\to\infty} \mathcal{E}(u - \varphi(u_n)) = 0$ by Proposition 3.16-(b). □

3.3 Fréchet derivative as a homeomorphism to the dual space

In many practical situations, the quotient normed space $\mathcal{F}/\mathcal{E}^{-1}(0)$ (equipped with the norm $\mathcal{E}^{1/p}$) becomes a Banach space (see Subsection 6.2). To state some basic properties of this Banach space, we recall the notion of *uniformly smoothness.*

Definition 3.19 (Uniformly smooth normed space)**.** Let $(\mathcal{X}, \|\cdot\|)$ be a normed space. The normed space $\mathcal{X}$ is said to be *uniformly smooth* if and only if it satisfies

$$\lim_{\tau\to 0} \tau^{-1} \sup\left\{ \frac{\|u+v\| + \|u-v\|}{2} - 1 \;\middle|\; \|u\| = 1, \|v\| = \tau \right\} = 0.$$

The following duality between uniform convexity and uniform smoothness is well known. (See also [BCL94, Lemma 5] for a quantitative version of this theorem.)

Theorem 3.20 (Day's duality theorem; see, e.g., [LT, Proposition 1.e.2])**.** *Let $\mathcal{X}$ be a Banach space. Then $\mathcal{X}$ is uniformly convex if and only if its dual space $\mathcal{X}^*$ is uniformly smooth.*

We also recall the notion of duality mapping and its fundamental results in the following proposition (see, e.g., [Miya, Definition 2.1, Lemmas 2.1 and 2.2]).

Proposition 3.21 (Duality mapping)**.** *Let $\mathcal{X}$ be a Banach space and let $\mathcal{X}^*$ be the dual space of $\mathcal{X}$. Let $\|\cdot\|_W$ be the norm of W for each $W \in \{\mathcal{X}, \mathcal{X}^*\}$. For $(x, f) \in \mathcal{X} \times \mathcal{X}^*$, we set $\langle x, f\rangle \coloneqq f(x)$. For $x \in \mathcal{X}$, define $F\colon \mathcal{X} \to 2^{\mathcal{X}^*}$ by*

$$F(x) \coloneqq \left\{ f \in \mathcal{X}^* \;\middle|\; \langle x, f\rangle = \|x\|_{\mathcal{X}}^2 = \|f\|_{\mathcal{X}^*}^2 \right\},$$

which is called the duality mapping of $\mathcal{X}$*. Then the following properties hold:*

(a) *$F(x) \neq \emptyset$ for any $x \in \mathcal{X}$.*
(b) *If $\mathcal{X}$ is reflexive, then $\bigcup_{x\in\mathcal{X}} F(x) = \mathcal{X}^*$.*
(c) *If $\mathcal{X}$ is strictly convex, i.e., $\|\lambda x + (1-\lambda)y\|_{\mathcal{X}} < \lambda\|x\|_{\mathcal{X}} + (1-\lambda)\|y\|_{\mathcal{X}}$ for any $\lambda \in (0,1)$ and any $x, y \in \mathcal{X}\setminus\{0\}$, then $\#(F(x)) = 1$ for any $x \in \mathcal{X}$.*

Now we can state a result on the dual space of $\mathcal{F}/\mathcal{E}^{-1}(0)$.

Theorem 3.22. *Assume that $(\mathcal{E},\mathcal{F})$ satisfies $(\mathrm{Cla})_p$ and that $\mathcal{F}/\mathcal{E}^{-1}(0)$ is a Banach space.*

(a) *The Banach space $\mathcal{F}/\mathcal{E}^{-1}(0)$ is uniformly convex and uniformly smooth. In particular, it is reflexive and its dual Banach spaces $\bigl(\mathcal{F}/\mathcal{E}^{-1}(0)\bigr)^*$ is also uniformly convex and uniformly smooth.*
(b) *The map $f \mapsto \mathcal{E}(f;\cdot)$ is a homeomorphism from $\mathcal{F}/\mathcal{E}^{-1}(0)$ to $\bigl(\mathcal{F}/\mathcal{E}^{-1}(0)\bigr)^*$. In particular, $\bigl(\mathcal{F}/\mathcal{E}^{-1}(0)\bigr)^* = \{\mathcal{E}(f;\,\cdot\,) \mid f \in \mathcal{F}\}$.*

Proof. For simplicity, set $\mathcal{X} \coloneqq \mathcal{F}/\mathcal{E}^{-1}(0)$ and $\|u\|_{\mathcal{X}} \coloneqq \mathcal{E}(u)^{1/p}$ for any $u \in \mathcal{X}$.

(a): The uniform convexity of $\mathcal{X}$ is immediate from Proposition 3.4, whence $\mathcal{X}$ is reflexive by the Milman–Pettis theorem. Also, we easily see from (3.18) that $\mathcal{X}$ is uniformly smooth. The same properties for $\mathcal{X}^*$ follow from Theorem 3.20.

(b): Let $u \in \mathcal{X}$ and define $\mathcal{A}(u) \coloneqq \mathcal{E}(u)^{2/p-1}\mathcal{E}(u;\,\cdot\,) \in \mathcal{X}^*$. (We define $\mathcal{A}(u) = 0$ if $\mathcal{E}(u) = 0$.) We will show that $\mathcal{A}\colon \mathcal{X} \to \mathcal{X}^*$ is a bijection. By (3.11), we have

$$\|\mathcal{A}(u)\|_{\mathcal{X}^*} = \mathcal{E}(u)^{2/p-1}\,\|\mathcal{E}(u;\,\cdot\,)\|_{\mathcal{X}^*} = \mathcal{E}(u)^{2/p-1+(p-1)/p} = \|u\|_{\mathcal{X}}\,.$$

Then $\langle u, \mathcal{A}(u)\rangle = \mathcal{E}(u)^{2/p} = \|u\|_{\mathcal{X}}^2 = \|\mathcal{A}(u)\|_{\mathcal{X}^*}^2$ and hence

$$\mathcal{A}(u) \in \{f \in \mathcal{X}^* \mid \langle u, f\rangle = \|u\|_{\mathcal{X}}^2 = \|f\|_{\mathcal{X}^*}^2\} = F(u),$$

where $F\colon \mathcal{X} \to \mathcal{X}^*$ is the duality mapping. We see from Proposition 3.21 and (a) that $\mathcal{A}\colon \mathcal{X} \to \mathcal{X}^*$ is a surjection. Note that the mapping $F^{-1}\colon \mathcal{X}^* \to \mathcal{X}^{**} = \mathcal{X}$ defined by $F^{-1}(f) = \{u \in \mathcal{X} \mid \langle u, f\rangle = \|u\|_{\mathcal{X}}^2 = \|f\|_{\mathcal{X}^*}^2\}$ for $f \in \mathcal{X}^*$ is the duality mapping from $\mathcal{X}^*$ to $\mathcal{X}$. By Proposition 3.21 and (a) again, we conclude that $\mathcal{A}$ is injective. The map $f \mapsto \mathcal{E}(f;\cdot)$ and its inverse are continuous by (3.11) and by (3.21) respectively. □

We also present a similar statement for $(\mathcal{F}, \|\cdot\|_{\mathcal{E},\alpha})$.

Corollary 3.23. *Let $\alpha \in (0,\infty)$. Assume that $\mathcal{F} \subseteq L^p(X,m)$, that $(\mathcal{E},\mathcal{F})$ satisfies $(\mathrm{Cla})_p$ and that $\mathcal{X}_\alpha \coloneqq (\mathcal{F}, \|\cdot\|_{\mathcal{E},\alpha})$ is a Banach space.*

(a) *The Banach space $\mathcal{X}_\alpha$ is uniformly convex and uniformly smooth. In particular, it is reflexive and its dual space $\mathcal{X}_\alpha^*$ is also uniformly convex and uniformly smooth.*
(b) *For each $f \in \mathcal{F}$, define a linear map $\Psi_{p,\alpha}^f\colon \mathcal{F} \to \mathbb{R}$ by*

$$\Psi_{p,\alpha}^f(g) \coloneqq \mathcal{E}(f;g) + \alpha\int_X \operatorname{sgn}(f)\,|f|^{p-1}\,g\,dm, \quad g \in \mathcal{F}. \tag{3.26}$$

Then the map $f \mapsto \Psi_{p,\alpha}^f$ is a homeomorphism from $\mathcal{X}_\alpha$ to $\mathcal{X}_\alpha^*$. In particular, $\mathcal{X}_\alpha^* = \{\Psi_{p,\alpha}^f \mid f \in \mathcal{F}\}$.

Proof. We define $\mathcal{E}_\alpha\colon \mathcal{F}\times\mathcal{F}\to\mathbb{R}$ by

$$\mathcal{E}_\alpha(u;v) \coloneqq \mathcal{E}(u;v) + \alpha\int_X \operatorname{sgn}(u)\,|u|^{p-1}\,v\,dm, \quad u,v\in\mathcal{F}.$$

and set $\mathcal{E}_\alpha(u) \coloneqq \mathcal{E}_\alpha(u;u) = \|u\|_{\mathcal{E},\alpha}^p$. Then $(\mathcal{E}_\alpha,\mathcal{F})$ is a p-energy form on (X,m) and it satisfies $(\mathrm{Cla})_p$ by Proposition 3.12. We have the desired result by applying Theorem 3.22 for $(\mathcal{E}_\alpha,\mathcal{F})$. □

3.4 Regularity and strong locality

In this subsection, in addition to the setting specified at the beginning of this section, similar to [FOT], we make the following topological assumptions[6]:

$$X \text{ is a locally compact metrizable space.} \tag{3.27}$$

$$m \text{ is a positive Radon measure on } X \text{ with full topological support.} \tag{3.28}$$

Note that (3.28) is equivalent to saying that $m(O)>0$ for any non-empty open subset O of X. Under this setting, the map $C(X)$ to $L^0(X,\mathcal{B},m)$, where $\mathcal{B}=\mathcal{B}(X)$, defined by taking $u\in C(X)$ to its m-equivalence class is injective and hence gives a canonical embedding of $C(X)$ into $L^0(X,m)$ as a subalgebra, and we will consider $C(X)$ as a subset of $L^0(X,m)$ through this embedding without further notice.

The following definitions are analogues of the notions in the theory of regular symmetric Dirichlet forms (see, e.g., [FOT, p. 6]).

Definition 3.24 (Core)**.** Let $\mathscr{C}$ be a subset of $\mathcal{F}\cap C_c(X)$.

(1) $\mathscr{C}$ is said to be a *core* of $(\mathcal{E},\mathcal{F})$ if and only if $\mathscr{C}$ is dense both in $(\mathcal{F},\|\cdot\|_{\mathcal{E},,1})$ and in $(C_c(X),\|\cdot\|_{\sup})$.

(2) A core $\mathscr{C}$ is said to be *special* if and only if $\mathscr{C}$ is a linear subspace of $\mathcal{F}\cap C_c(X)$, $\mathscr{C}$ is a dense subalgebra of $(C_c(X),\|\cdot\|_{\sup})$, and for any compact subset K of X and any relatively compact open subset G of X with $K\subseteq G$, there exists $\varphi\in\mathscr{C}$ such that $\varphi\geq 0$, $\varphi=1$ on K and $\varphi=0$ on $X\setminus G$.

Definition 3.25 (Regularity)**.** We say that $(\mathcal{E},\mathcal{F})$ is *regular* if and only if there exists a core $\mathscr{C}$ of $(\mathcal{E},\mathcal{F})$.

We can show the following result on regular p-energy forms, which is an analogue of [FOT, Exercise 1.4.1].

Proposition 3.26. *Suppose that $(\mathcal{E},\mathcal{F})$ is regular and that $\mathcal{F}$ satisfies the following properties:*

$$u^+\wedge 1\in\mathcal{F} \quad \textit{for any } u\in\mathcal{F}, \tag{3.29}$$

$$uv\in\mathcal{F} \quad \textit{for any } uv\in\mathcal{F}\cap C_b(X). \tag{3.30}$$

Then $\mathcal{F}\cap C_c(X)$ is a special core of $(\mathcal{E},\mathcal{F})$.

[6] We do not assume that X is separable unlike [FOT, (1,1,7)].

Proof. It is clear that $\mathcal{F}\cap C_c(X)$ is a core of $(\mathcal{E},\mathcal{F})$. By (3.30), $\mathcal{F}\cap C_c(X)$ is a subalgebra of $C_c(X)$. Let K be a compact subset of X and G be a relatively compact open subset G of X with $K \subseteq G$. By Urysohn's lemma, there exists $\varphi_0 \in C_c(X)$ such that $\varphi_0 = 2$ on K and $\varphi_0 = 0$ on $X \setminus G$. Let $\varepsilon \in (0,1/2)$. Fix $\psi \in \mathcal{F}\cap C_c(X)$ satisfying $\psi = 1$ on $\overline{G}^X$, which exists by the regularity of $(\mathcal{E},\mathcal{F})$, the locally compactness of X and (3.29). Since $\mathcal{F}\cap C_c(X)$ is a core of $(\mathcal{E},\mathcal{F})$, there exists $\widetilde{\varphi} \in \mathcal{F}\cap C_c(X)$ such that $\|\varphi_0 - \widetilde{\varphi}\|_{\sup} < \varepsilon$. Now we define $\varphi \in C_c(X)$ by $\varphi \coloneqq (\widetilde{\varphi} - \varepsilon\psi)^+ \wedge 1$. (Note that $\mathrm{supp}_X[\varphi]$ is compact since $\overline{G}^X$ is compact.) Then $\varphi \in \mathcal{F}\cap C_c(X)$ by (3.29). Clearly, $\varphi = 1$ on K and $\varphi = 0$ on $X \setminus G$, so the proof is completed. □

The proposition above ensures when there exist *cutoff* functions in $\mathcal{F}$. We also introduce the following condition stating the existence of cutoff functions in a weaker sense.

Definition 3.27. We say that a p-energy form $(\mathcal{E},\mathcal{F})$ on (X,m) satisfies the property $(\mathrm{CF})_m$[7] if and only if, for any compact subset K of X and any open subset U of X with $K \subseteq U$, there exists $\varphi \in \mathcal{F}\cap L^\infty(X,m)$ such that $\varphi(x) = 1$ for m-a.e. $x \in K$ and $\varphi(x) = 0$ for m-a.e. $x \in X \setminus U$.

Next we introduce two formulations of the notion of strong locality for $(\mathcal{E},\mathcal{F})$.

Definition 3.28 (Strong locality)**.** (1) We say that $(\mathcal{E},\mathcal{F})$ has the strongly local property (SL1) if and only if, for any $f_1, f_2, g \in \mathcal{F}$ with either $\mathrm{supp}_m[f_1 - \alpha_1]$ or $\mathrm{supp}_m[f_2 - \alpha_2]$ compact and $\mathrm{supp}_m[f_1 - \alpha_1] \cap \mathrm{supp}_m[f_2 - \alpha_2] = \emptyset$ for some $\alpha_1, \alpha_2 \in \mathcal{E}^{-1}(0)$,

$$\mathcal{E}(f_1 + f_2 + g) + \mathcal{E}(g) = \mathcal{E}(f_1 + g) + \mathcal{E}(f_2 + g). \tag{3.31}$$

(2) Suppose that $(\mathcal{E},\mathcal{F})$ satisfies $(\mathrm{Cla})_p$. We say that $(\mathcal{E},\mathcal{F})$ has the strongly local property (SL2) if and only if, for any $f_1, f_2, g \in \mathcal{F}$ with either $\mathrm{supp}_m[f_1 - f_2 - \alpha]$ or $\mathrm{supp}_m[g - \beta]$ compact and $\mathrm{supp}_m[f_1 - f_2 - \alpha] \cap \mathrm{supp}_m[g - \beta] = \emptyset$ for some $\alpha, \beta \in \mathcal{E}^{-1}(0)$,

$$\mathcal{E}(f_1; g) = \mathcal{E}(f_2; g). \tag{3.32}$$

In the following propositions, we collect basic results about (SL1) and (SL2).

Proposition 3.29. *Assume that $(\mathcal{E},\mathcal{F})$ satisfies $(\mathrm{Cla})_p$.*

(a) *If $(\mathcal{E},\mathcal{F})$ satisfies (SL1), then for any $f_1, f_2, g \in \mathcal{F}$ with either $\mathrm{supp}_m[f_1 - \alpha_1]$ or $\mathrm{supp}_m[f_2 - \alpha_2]$ compact and $\mathrm{supp}_m[f_1 - \alpha_1] \cap \mathrm{supp}_m[f_2 - \alpha_2] = \emptyset$ for some $\alpha_1, \alpha_2 \in \mathcal{E}^{-1}(0)$,*

$$\mathcal{E}(f_1 + f_2; g) = \mathcal{E}(f_1; g) + \mathcal{E}(f_2; g). \tag{3.33}$$

(b) *If $(\mathcal{E},\mathcal{F})$ satisfies (SL2), then for any $f_1, f_2, g \in \mathcal{F}$ with either $\mathrm{supp}_m[f_1 - f_2 - \alpha]$ or $\mathrm{supp}_m[g - \beta]$ compact and $\mathrm{supp}_m[f_1 - f_2 - \alpha] \cap \mathrm{supp}_m[g - \beta] = \emptyset$ for some $\alpha, \beta \in \mathcal{E}^{-1}(0)$,*

$$\mathcal{E}(g; f_1) = \mathcal{E}(g; f_2). \tag{3.34}$$

[7]We can consider several versions of this condition such as a version requiring $\varphi \in \mathcal{F}\cap C(K)$ in addition. Note that $(\mathrm{CF})_m$ holds if $(\mathcal{E},\mathcal{F})$ admits a special core.

Proof. (a): Note that (3.31) with $g = 0$ implies that $\mathcal{E}(f_1 + f_2) = \mathcal{E}(f_1) + \mathcal{E}(f_2)$. For any $t \in (0, \infty)$, we have from (3.31) that

$$\frac{\mathcal{E}(f_1 + f_2 + tg) - \mathcal{E}(f_1 + f_2)}{t} + t^{p-1}\mathcal{E}(g) = \frac{\mathcal{E}(f_1 + tg) - \mathcal{E}(f_1)}{t} + \frac{\mathcal{E}(f_2 + tg) - \mathcal{E}(f_2)}{t}.$$

We obtain (3.33) by letting $t \downarrow 0$ in this equality.

(b): Since $\mathcal{E}(g; \cdot\,)$ is linear by Theorem 3.6, it suffices to prove $\mathcal{E}(g; f_1 - f_2) = 0$, which follows from (3.32) with $g, 0, f_1 - f_2$ in place of f_1, f_2, g. □

Proposition 3.30. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies* (Cla)$_p$*.*

(a) *If $(\mathcal{E}, \mathcal{F})$ satisfies* (SL1)*, then $(\mathcal{E}, \mathcal{F})$ also satisfies* (SL2)*.*

(b) *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies* (SL2) *and the following three conditions:*

$$uv \in \mathcal{F} \cap L^\infty(X, m) \text{ for any } u, v \in \mathcal{F} \cap L^\infty(X, m). \tag{3.35}$$

$$\text{For any } u \in \mathcal{F},\ u_n := (-n) \vee u \wedge n \in \mathcal{F} \text{ and } \lim_{n \to \infty} \mathcal{E}(u - u_n) = 0. \tag{3.36}$$

$$(\mathcal{E}, \mathcal{F}) \text{ satisfies } (\mathrm{CF})_m. \tag{3.37}$$

Then $(\mathcal{E}, \mathcal{F})$ satisfies (SL1)*.*

Proof. (a): Let $f_1, f_2, g \in \mathcal{F}$ and $\alpha_1, \alpha_2 \in \mathcal{E}^{-1}(0)$ with either $\mathrm{supp}_m[f_1 - f_2 - \alpha]$ or $\mathrm{supp}_m[g - \beta]$ compact and $\mathrm{supp}_m[f_1 - f_2 - \alpha] \cap \mathrm{supp}_m[g - \beta] = \emptyset$. Let $t \in (0, 1)$. By (3.31) with $t(f_1 - f_2), g, 0$ in place of f_1, f_2, g, we have

$$\mathcal{E}(t(f_1 - f_2) + g) = \mathcal{E}(t(f_1 - f_2)) + \mathcal{E}(g),$$

whence

$$\lim_{t \downarrow 0} \frac{\mathcal{E}(g + t(f_1 - f_2)) - \mathcal{E}(g)}{t} = \lim_{t \downarrow 0} t^{p-1}\mathcal{E}(f_1 - f_2) = 0.$$

Since $\mathcal{E}(g; \cdot\,)$ is linear by Theorem 3.6, we get $\mathcal{E}(g; f_1) = \mathcal{E}(g; f_2)$. Similarly, by (3.31) with $f_2 - f_1, tg, f_1$ in place of f_1, f_2, g,

$$\mathcal{E}\bigl((f_2 - f_1) + tg + f_1\bigr) + \mathcal{E}(f_1) = \mathcal{E}\bigl((f_2 - f_1) + f_1\bigr) + \mathcal{E}(tg + f_1),$$

which implies $\mathcal{E}(f_1; g) = \mathcal{E}(f_2; g)$.

(b): We first consider the case $g \in \mathcal{F} \cap L^\infty(X, m)$. Let $f_1, f_2 \in \mathcal{F}$ and $\alpha_1, \alpha_2 \in \mathcal{E}^{-1}(0)$ with either $\mathrm{supp}_m[f_1 - \alpha_1]$ or $\mathrm{supp}_m[f_2 - \alpha_2]$ compact and $\mathrm{supp}_m[f_1 - \alpha_1] \cap \mathrm{supp}_m[f_2 - \alpha_2] = \emptyset$. We assume that $\mathrm{supp}_m[f_1 - \alpha_1]$ is compact since both cases are similar. Let U be an open neighborhood of $\mathrm{supp}_m[f_1 - \alpha_1]$ such that $U \subseteq X \setminus \mathrm{supp}_m[f_2 - \alpha_2]$. By (3.37) and the locally compactness of K, there exists $\varphi \in \mathcal{F} \in L^\infty(X, m)$ such that $\varphi(x) = 1$ for m-a.e. $x \in U$, $\mathrm{supp}_m[\varphi]$ is compact and $\mathrm{supp}_m[\varphi] \cap \mathrm{supp}_m[f_2 - \alpha_2] = \emptyset$. Note that $\varphi g \in \mathcal{F}$ by (3.35). Then we see from (SL2) that

$$\begin{aligned}
\mathcal{E}(f_1 + f_2 + g) + \mathcal{E}(g) &= \mathcal{E}(f_1 + f_2 + g; f_1) + \mathcal{E}(f_1 + f_2 + g; f_2) + \mathcal{E}(f_1 + f_2 + g; g) + \mathcal{E}(g) \\
&\overset{\text{(SL2)}}{=} \mathcal{E}(f_1 + g; f_1) + \mathcal{E}(f_2 + g; f_2) + \mathcal{E}(f_1 + f_2 + g; g) + \mathcal{E}(g)
\end{aligned}$$

$$= \mathcal{E}(f_1 + g; f_1) + \mathcal{E}(f_2 + g; f_2) \\ + \mathcal{E}(f_1 + f_2 + g; (1 - \varphi)g) + \mathcal{E}(f_1 + f_2 + g; \varphi g) + \mathcal{E}(g). \tag{3.38}$$

Since $\mathrm{supp}_m[\varphi g]$ and $\mathrm{supp}_m[f_1 - \alpha_1]$ are compact, $\mathrm{supp}_m[f_1 - \alpha_1] \cap \mathrm{supp}_m[(1 - \varphi)g] = \emptyset$ and $\mathrm{supp}_m[f_2 - \alpha_2] \cap \mathrm{supp}_m[\varphi g] = \emptyset$, we have the following equalities by (SL2):

$$\mathcal{E}(f_1 + f_2 + g; (1 - \varphi)g) = \mathcal{E}(f_2 + g; (1 - \varphi)g).$$
$$\mathcal{E}(f_1 + f_2 + g; \varphi g) = \mathcal{E}(f_1 + g; \varphi g).$$
$$\mathcal{E}(g) = \mathcal{E}(g; (1 - \varphi)g) + \mathcal{E}(g; \varphi g) = \mathcal{E}(f_1 + g; (1 - \varphi)g) + \mathcal{E}(f_2 + g; \varphi g).$$

By combining these equalities and (3.38), we obtain

$$\mathcal{E}(f_1 + f_2 + g) + \mathcal{E}(g) = \mathcal{E}(f_1 + g; f_1) + \mathcal{E}(f_2 + g; f_2) + \mathcal{E}(f_1 + g; g) + \mathcal{E}(f_2 + g; g) \\ = \mathcal{E}(f_1 + g) + \mathcal{E}(f_2 + g),$$

which proves (SL1) in the case $g \in \mathcal{F} \cap L^\infty(X, m)$.

Lastly, we prove (SL1) without assuming the boundedness of g. Let $g \in \mathcal{F}$ and set $g_n := (-n) \vee (g \wedge n)$, $n \in \mathbb{N}$. Then $g_n \in \mathcal{F}$ by (3.36), and the statement proved in the previous paragraph yields that

$$\mathcal{E}(f_1 + f_2 + g_n) + \mathcal{E}(g_n) = \mathcal{E}(f_1 + g_n) + \mathcal{E}(f_2 + g_n)$$

for any $n \in \mathbb{N}$. Thanks to (3.36) and the triangle inequality for $\mathcal{E}^{1/p}$, we obtain the desired equality (3.32) by letting $n \to \infty$ in the equality above. □

4 p-Energy measures and their basic properties

In this section, we discuss p-energy measures dominated by a p-energy form. Similar to the case of p-energy forms, we will introduce two-variable versions of p-energy measures and prove their basic properties.

As in the previous section, in this section, we fix $p \in (1, \infty)$, a measure space $(X, \mathcal{B}, m)$ and a p-energy form $(\mathcal{E}, \mathcal{F})$ on (X, m) with $\mathcal{F} \subseteq L^0(X, m)$.

4.1 p-Energy measures and p-Clarkson's inequalities

In this subsection, we also assume the existence of a family of finite measures $\{\Gamma\langle f \rangle\}_{f \in \mathcal{F}}$ on $(X, \mathcal{B})$ whose definition is as follows.

Definition 4.1 (p-Energy measures dominated by a p-energy form)**.** Let $\{\Gamma\langle f \rangle\}_{f \in \mathcal{F}}$ be a family of measures on $(X, \mathcal{B})$. We say that $\{\Gamma\langle f \rangle\}_{f \in \mathcal{F}}$ are *p-energy measures dominated by* $(\mathcal{E}, \mathcal{F})$ if and only if the following hold:

$(\mathrm{EM1})_p$ $\Gamma\langle f \rangle(X) \le \mathcal{E}(f)$ for any $f \in \mathcal{F}$.

$(\mathrm{EM2})_p$ $\Gamma\langle \cdot \rangle(A)^{1/p}$ is a seminorm on $\mathcal{F}$ for any $A \in \mathcal{B}$.

We then see that $(\Gamma\langle\,\cdot\,\rangle(A), \mathcal{F})$ is a p-energy form on (X, m) for each $A \in \mathcal{B}$ by (EM2)$_p$.

We say that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies p-Clarkson's inequality, (Cla)$_p$ for short, if and only if $(\Gamma\langle\,\cdot\,\rangle(A), \mathcal{F})$ satisfies (Cla)$_p$ for any $A \in \mathcal{B}$, i.e., for any $f, g \in \mathcal{F}$,

$$\begin{cases} \Gamma\langle f+g\rangle(A)^{\frac{1}{p-1}} + \Gamma\langle f-g\rangle(A)^{\frac{1}{p-1}} \leq 2\bigl(\Gamma\langle f\rangle(A) + \Gamma\langle g\rangle(A)\bigr)^{\frac{1}{p-1}} & \text{if } p \in (1,2], \\ \Gamma\langle f+g\rangle(A) + \Gamma\langle f-g\rangle(A) \leq 2\bigl(\Gamma\langle f\rangle(A)^{\frac{1}{p-1}} + \Gamma\langle g\rangle(A)^{\frac{1}{p-1}}\bigr)^{p-1} & \text{if } p \in (2,\infty). \end{cases} \tag{Cla$_p$}$$

We also say that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies the generalized p-contraction property, (GC)$_p$ for short, if and only if $(\Gamma\langle\,\cdot\,\rangle(A), \mathcal{F})$ satisfies (GC)$_p$ for any $A \in \mathcal{B}$.

Example 4.2. (1) Consider the same setting as in Example 3.10-(1). Then the measures

$$\Gamma\langle f\rangle(A) \coloneqq \int_A |\nabla f(x)|^p\,dx \quad \text{for } f \in W^{1,p}(\Omega) \text{ and } A \in \mathcal{B}(\mathbb{R}^D) \text{ with } A \subseteq \Omega,$$

are easily seen to be p-energy measures dominated by $\mathcal{E}(f) = \int_\Omega |\nabla f(x)|^p\,dx$ satisfying (EM1)$_p$ and (EM2)$_p$. Similar to Example 3.10-(1), one can show (GC)$_p$ for $\{\Gamma\langle f\rangle\}_{f\in W^{1,p}(\Omega)}$ by following an argument in the proof of Theorem A.19. Recall that $\mathcal{E}(f;g) = \int_\Omega |\nabla f(x)|^{p-2}\langle\nabla f(x), \nabla g(x)\rangle_{\mathbb{R}^D}\,dx$. Then we can see that, by the Leibniz and chain rules for ∇, for any $u, \varphi \in W^{1,p}(\Omega) \cap C^1(\Omega)$,

$$\int_\Omega \varphi\,d\Gamma\langle u\rangle = \mathcal{E}(u; u\varphi) - \left(\frac{p-1}{p}\right)^{p-1} \mathcal{E}\bigl(|u|^{\frac{p}{p-1}}; \varphi\bigr). \tag{4.1}$$

(2) Although p-energies are constructed on compact metric spaces [Kig23, MS23+], we do not know how to construct the associated p-energy measures because of the lack of the density "$|\nabla u(x)|^p$". (As described in (3) below, the theory of Dirichlet forms presents 2-energy measures $\{\mu_{\langle u\rangle}\}_{u\in\mathcal{F}_2}$ associated with a given nice Dirichlet form $(\mathcal{E}_2, \mathcal{F}_2)$. On a large class of self-similar sets, it is known that $\mu_{\langle u\rangle}$ is mutually singular with respect to the natural Hausdorff measure of the underlying fractal [Hin05, KM20].) In the case of self-similar sets with suitable assumptions, *self-similar p-energy forms* are constructed in [CGQ22, Kig23, MS23+, Shi24], and we can introduce p-energy measures satisfying (EM1)$_p$, (EM2)$_p$ and (GC)$_p$ by using the self-similarity of p-energy forms. See Section 5 for details.

In [KS.a], under suitable assumptions, the authors construct a good p-energy form $\mathcal{E}_p^{\mathrm{KS}}$, which is called a *Korevaar–Shoen p-energy form*, on a locally compact separable metric space (X, d) equipped with a σ-finite Borel measure m with full topological support. As an advantage of $\mathcal{E}_p^{\mathrm{KS}}$, the right-hand side of (4.1) with $\mathcal{E}_p^{\mathrm{KS}}$ in place of $\mathcal{E}$ can be extended to a bounded positive linear functional in $\varphi \in C_c(X)$ and the p-energy measure $\Gamma_p^{\mathrm{KS}}\langle u\rangle$ associated with $\mathcal{E}_p^{\mathrm{KS}}$ is constructed as the unique Radon measure corresponding to this functional through the Riesz–Markov–Kakutani representation theorem. A notable fact is that this approach does not rely on the self-similarity of the underlying space or of the p-energy form. In [KS.a, Sections 3 and 4], basic properties for $\Gamma_p^{\mathrm{KS}}\langle\,\cdot\,\rangle$ like (EM1)$_p$, (EM2)$_p$ and (GC)$_p$ are also shown.

(3) The case $p = 2$ is very special thanks to the theory of symmetric Dirichlet forms. (See [FOT, Section 3.2] for details on 2-energy measures associated with regular symmetric Dirichlet forms.) If $(\mathscr{E}, D(\mathscr{E}))$ is a regular strongly local Dirichlet form on $L^2(X, m)$, where X is a locally compact separable metrizable space and m is a Radon measure on X with full topological support (see [FOT, (1.1.7)]), then $\mathscr{E}(u) := \mathscr{E}(u, u)$ is a 2-energy form on (X, m) and it satisfies (GC)$_2$ (see Proposition A.2). In addition, the Dirichlet form theory provides us the associated 2-energy measures $\{\mu_{\langle u\rangle}\}_{u\in D(\mathscr{E})}$ through the following formula[8]:

$$\int_X \varphi\, d\mu_{\langle u\rangle} = \mathscr{E}(u, u\varphi) - \frac{1}{2}\mathscr{E}(u^2, \varphi) \quad \text{for any } \varphi \in D(\mathscr{E}) \cap C_c(X). \tag{4.2}$$

(Recall (4.1).) We easily see that $\{\mu_{\langle u\rangle}\}_{u\in D(\mathscr{E})}$ satisfies (EM1)$_2$ and the parallelogram law, which implies (EM2)$_2$ and (Cla)$_2$. We can also verify (GC)$_2$ for $\{\mu_{\langle u\rangle}\}_{u\in D(\mathscr{E})}$ (see Proposition A.14). In the framework of [Kuw24] (see also Definition A.17), we can introduce p-energy measures satisfying (EM1)$_p$, (EM2)$_p$ and (GC)$_p$ by setting $\Gamma\langle u\rangle(A) := \int_A \Gamma_\mu(u)^{\frac{p}{2}}\, d\mu$, where μ is a $\mathscr{E}$-dominant measure; in particular $\mu_{\langle u\rangle} \ll \mu$, and $\Gamma_\mu := \frac{d\mu_{\langle u\rangle}}{d\mu}$. See Theorem A.19 for a proof of (GC)$_p$ for these p-energy measures.

(4) Let g_u be the minimal p-weak upper gradient of $u \in N^{1,p}(X, m)$, where $N^{1,p}(X, m) := \{u \in L^p(X, m) \mid g_u \in L^p(X, m)\}$ is the Newton-Sobolev space (see [HKST, Section 7.1]). Then $\Gamma\langle u\rangle(A) := \int_A g_u^p\, dm$ defines p-energy measures satisfying (EM1)$_p$ and (EM2)$_p$. Indeed, we have (EM2)$_p$ by [HKST, (6.3.18)]. However, (Cla)$_p$ for these measures is unclear because of the lack of the linearity of $u \mapsto g_u$.

The same argument as in Proposition 3.5 yields the following result.

Proposition 4.3. *Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies (Cla)$_p$. Then, for any $f, g \in \mathcal{F}$ and any $A \in \mathcal{B}$,*

$$\begin{aligned} &\Gamma\langle f+g\rangle(A) + \Gamma\langle f-g\rangle(A) - 2\Gamma\langle f\rangle(A) \\ &\quad\le \begin{cases} 2\Gamma\langle g\rangle(A) & \text{if } p \in (1,2], \\ 2(p-1)\Bigl[\Gamma\langle f\rangle(A)^{\frac{1}{p-1}} + \Gamma\langle g\rangle(A)^{\frac{1}{p-1}}\Bigr]^{p-2}\Gamma\langle g\rangle(A)^{\frac{1}{p-1}} & \text{if } p \in (2,\infty). \end{cases} \end{aligned} \tag{4.3}$$

In particular, $\mathbb{R} \ni t \mapsto \Gamma\langle f+tg\rangle(A) \in [0,\infty)$ is differentiable and for any $s \in \mathbb{R}$,

$$\lim_{\delta\downarrow 0} \sup_{\substack{A\in\mathcal{B}, g\in\mathcal{F};\\ \mathcal{E}(g)\le 1}} \left| \frac{\Gamma\langle f+(s+\delta)g\rangle(A) - \Gamma\langle f+sg\rangle(A)}{\delta} - \frac{d}{dt}\Gamma\langle f+tg\rangle(A)\Big|_{t=s} \right| = 0. \tag{4.4}$$

Definition 4.4. Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies (Cla)$_p$. Let $f, g \in \mathcal{F}$. Define $\Gamma\langle f; g\rangle \colon \mathcal{B} \to \mathbb{R}$ by

$$\Gamma\langle f; g\rangle(A) := \frac{1}{p}\frac{d}{dt}\Gamma\langle f+tg\rangle(A)\Big|_{t=0} \qquad \text{for } A \in \mathcal{B}(X), \tag{4.5}$$

which exists by Proposition 4.3.

[8] Precisely, the formula (4.2) is valid for $u \in D(\mathscr{E}) \cap L^\infty(X, m)$. We can extend it to any $u \in D(\mathscr{E})$ by considering the limit of $(u \wedge n) \vee (-n)$ as $n \to \infty$

The following properties of $\Gamma\langle f; g\rangle$ can be shown in a similar way as Theorem 3.6.

Theorem 4.5. *Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies* (Cla)$_p$. *Let $A \in \mathcal{B}$. Then $\Gamma\langle f; \cdot\rangle(A)$ is the Fréchet derivative of $\Gamma\langle\,\cdot\,\rangle(A)\colon \mathcal{F}/\mathcal{E}^{-1}(0) \to [0,\infty)$ at $f \in \mathcal{F}$. In particular, the map $\Gamma\langle f; \cdot\rangle(A)\colon \mathcal{F} \to \mathbb{R}$ is linear, $\Gamma\langle f; f\rangle(A) = \Gamma\langle f\rangle(A)$ and $\Gamma\langle f; h\rangle(A) = 0$ if $h \in \mathcal{F}$ satisfies $\Gamma\langle h\rangle(A) = 0$. Moreover, for any $f, f_1, f_2, g \in \mathcal{F}$ and $a \in \mathbb{R}$, the following hold:*

$$\mathbb{R} \ni t \mapsto \Gamma\langle f + tg; g\rangle(A) \in \mathbb{R} \text{ is strictly increasing if and only if } \Gamma\langle g\rangle(A) > 0. \tag{4.6}$$

$$\Gamma\langle af; g\rangle = \operatorname{sgn}(a)\,|a|^{p-1}\,\Gamma\langle f; g\rangle, \quad \Gamma\langle f + h; g\rangle(A) = \Gamma\langle f; g\rangle(A) \text{ if } \Gamma\langle h\rangle(A) = 0. \tag{4.7}$$

$$|\Gamma\langle f; g\rangle(A)| \le \Gamma\langle f\rangle(A)^{(p-1)/p}\Gamma\langle g\rangle(A)^{1/p}. \tag{4.8}$$

$$|\Gamma\langle f_1; g\rangle(A) - \Gamma\langle f_2; g\rangle(A)| \le C_p\bigl(\Gamma\langle f_1\rangle(A) \vee \Gamma\langle f_2\rangle(A)\bigr)^{\frac{p-1-\alpha_p}{p}}\Gamma\langle f_1 - f_2\rangle(A)^{\frac{\alpha_p}{p}}\Gamma\langle g\rangle(A)^{\frac{1}{p}}, \tag{4.9}$$

where α_p, C_p are the same as in Theorem 3.6.

The set function $\Gamma\langle f; g\rangle$ is a signed measure as shown in the following proposition.

Proposition 4.6. *Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies* (Cla)$_p$. *For any $f, g \in \mathcal{F}$, the set function $\Gamma\langle f; g\rangle$ is a signed measure on $(X, \mathcal{B})$. Moreover, for any $\mathcal{B}$-measurable function $\varphi\colon X \to [0,\infty)$ with $\|\varphi\|_{\sup} < \infty$, $\int_X \varphi\, d\Gamma\langle\,\cdot\,\rangle\colon \mathcal{F}/\mathcal{E}^{-1}(0) \to \mathbb{R}$ is Fréchet differentiable and has the same properties as those of $\Gamma\langle\,\cdot\,\rangle$ in Theorem 4.5 with "$\Gamma\langle g\rangle(A) > 0$" in (4.6) replaced by "$\int_X \varphi\, d\Gamma\langle g\rangle > 0$", and for any $f, g \in \mathcal{F}$,*

$$\int_X \varphi\, d\Gamma\langle f; g\rangle = \frac{1}{p}\,\frac{d}{dt}\int_X \varphi\, d\Gamma\langle f + tg\rangle\Big|_{t=0}. \tag{4.10}$$

Proof. The equalities $\Gamma\langle f; g\rangle(\emptyset) = 0$ and $|\Gamma\langle f; g\rangle(X)| = |\mathcal{E}(f; g)| < \infty$ are clear from the definition. We will show the countable additivity of $\Gamma\langle f; g\rangle$. The finite additivity of $\Gamma\langle f; g\rangle$ is obvious. Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}$ be a family of disjoint measurable sets. Set $B_N := \bigcup_{n=N+1}^{\infty} A_n$ for each $N \in \mathbb{N}$. Then we see that

$$\begin{aligned}\left|\Gamma\langle f; g\rangle\left(\bigcup_{n\in\mathbb{N}} A_n\right) - \sum_{n=1}^{N}\Gamma\langle f; g\rangle(A_n)\right| &= |\Gamma\langle f; g\rangle(B_N)| \\ &\overset{(4.8)}{\le} \Gamma\langle f\rangle(B_N)^{(p-1)/p}\Gamma\langle g\rangle(B_N)^{1/p} \xrightarrow[N\to\infty]{} 0,\end{aligned}$$

which shows that $\Gamma\langle f; g\rangle$ is a signed measure on $(X, \mathcal{B})$.

The other properties except for (4.10) can be proved by following the arguments in the proof of Theorem 3.6, so we shall prove (4.10). By the finite additivity of $\int_X \varphi\, d\Gamma\langle f; g\rangle$ and $\frac{1}{p}\frac{d}{dt}\int_X \varphi\, d\Gamma\langle f + tg\rangle\big|_{t=0}$ in φ, we can assume that $\varphi \ge 0$. Let $s_n = \sum_{k=1}^{l_n} a_k \mathbb{1}_{A_k}$ with $a_k \ge 0$ and $A_k \in \mathcal{B}$ be a sequence of simple functions so that $s_n \uparrow \varphi$ m-a.e. as $n \to \infty$. Then we immediately have (4.10) with $\varphi = s_n$. Since $\lim_{n\to\infty}\int_X s_n\, d\Gamma\langle f; g\rangle = \int_X \varphi\, d\Gamma\langle f; g\rangle$ by the dominated convergence theorem, it suffices to prove

$$\lim_{n\to\infty}\frac{d}{dt}\int_X s_n\, d\Gamma\langle f + tg\rangle\Big|_{t=0} = \frac{d}{dt}\int_X \varphi\, d\Gamma\langle f + tg\rangle\Big|_{t=0}. \tag{4.11}$$

Since (3.15) with $\int_X \varphi\, d\Gamma\langle\,\cdot\,\rangle$ in place of $\mathcal{E}$ holds by the fact that $(\int_X \varphi\, d\Gamma\langle\,\cdot\,\rangle, \mathcal{F})$ is a p-energy form on (X, m), we know that for any $\mathcal{B}$-measurable function $\psi\colon X \to [0,\infty)$ with $\|\psi\|_{\sup} < \infty$

$$\left|\frac{d}{dt}\int_X \psi\, d\Gamma\langle f + tg\rangle\Big|_{t=0}\right| \le \left(\int_X \psi\, d\Gamma\langle f\rangle\right)^{(p-1)/p}\left(\int_X \psi\, d\Gamma\langle g\rangle\right)^{1/p}. \tag{4.12}$$

By combining (4.12) with $\psi = \varphi - s_n$ and the dominated convergence theorem, we obtain (4.11). □

Remark 4.7. As mentioned in the introduction, a signed measure corresponding to $\Gamma\langle f; g\rangle$ is discussed in [BV05, Section 5] under some non-trivial assumptions, which have not been verified for fractals like the Sierpiński gasket and the Sierpiński carpet in the literature.

The following proposition gives a Hölder-type estimate with respect to the total variation measure $|\Gamma\langle f; g\rangle|$.

Proposition 4.8. *Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies (Cla)$_p$. For any $f, g \in \mathcal{F}$ and any $\mathcal{B}$-measurable functions $\varphi, \psi\colon X \to [0,\infty]$,*

$$\int_X \varphi\psi\, d\,|\Gamma\langle f; g\rangle| \le \left(\int_X \varphi^{\frac{p}{p-1}}\, d\Gamma\langle f\rangle\right)^{(p-1)/p}\left(\int_X \psi^p\, d\Gamma\langle g\rangle\right)^{1/p}. \tag{4.13}$$

Proof. Let $X = \mathcal{P} \sqcup \mathcal{N}$ be the Hahn decomposition with respect to $\Gamma\langle f; g\rangle$ such that $\Gamma\langle f; g\rangle(A) \ge 0$ for any Borel set $A \subseteq \mathcal{P}$ and $\Gamma\langle f; g\rangle(A) \le 0$ for any Borel set $A \subseteq \mathcal{N}$. Then the total variation measure $|\Gamma\langle f; g\rangle|$ is given by

$$|\Gamma\langle f; g\rangle|\,(A) = \Gamma\langle f; g\rangle(\mathcal{P}\cap A) - \Gamma\langle f; g\rangle(\mathcal{N}\cap A) \quad \text{for any } A \in \mathcal{B}.$$

Therefore, by (4.8),

$$\begin{aligned}
|\Gamma\langle f; g\rangle|\,(A) &\le \Gamma\langle f\rangle(\mathcal{P}\cap A)^{(p-1)/p}\Gamma\langle g\rangle(\mathcal{P}\cap A)^{1/p} + \Gamma\langle f\rangle(\mathcal{N}\cap A)^{(p-1)/p}\Gamma\langle g\rangle(\mathcal{N}\cap A)^{1/p} \\
&\le \bigl(\Gamma\langle f\rangle(\mathcal{P}\cap A) + \Gamma\langle f\rangle(\mathcal{N}\cap A)\bigr)^{(p-1)/p}\bigl(\Gamma\langle g\rangle(\mathcal{P}\cap A) + \Gamma\langle g\rangle(\mathcal{N}\cap A)\bigr)^{1/p} \\
&= \Gamma\langle f\rangle(A)^{(p-1)/p}\Gamma\langle g\rangle(A)^{1/p},
\end{aligned} \tag{4.14}$$

where we used Hölder's inequality in the third line.

Now we prove (4.13). First, we consider the case that φ and ψ are non-negative simple functions, that is,

$$\varphi = \sum_{k=1}^{N_1} \widetilde{a}_k \mathbb{1}_{A_k}, \quad \psi = \sum_{k=1}^{N_2} \widetilde{b}_k \mathbb{1}_{B_k}, \text{ where } \widetilde{a}_k, \widetilde{b}_k \in [0,\infty) \text{ and } A_k, B_k \in \mathcal{B}.$$

Then we can assume that there exist $N \in \mathbb{N}$, $\{a_k\}_{k=1}^N, \{b_k\}_{k=1}^N \subseteq [0,\infty)$ and a disjoint family of measurable sets $\{E_k\}_{k=1}^N \subseteq \mathcal{B}$ such that $\varphi = \sum_{k=1}^N a_k \mathbb{1}_{E_k}$ and $\psi = \sum_{k=1}^N b_k \mathbb{1}_{E_k}$. Since $uv = \sum_{k=1}^N a_k b_k \mathbb{1}_{E_k}$, a combination of (4.14) and Hölder's inequality yields

$$\int_X \varphi\psi\, d\,|\Gamma\langle f; g\rangle| = \sum_{k=1}^N a_k b_k\, |\Gamma\langle f; g\rangle(E_k)|$$

$$\leq \left(\sum_{k=1}^{N} a_k^{p/(p-1)} \Gamma\langle f\rangle(E_k)\right)^{(p-1)/p} \left(\sum_{k=1}^{N} b_k^p \Gamma\langle g\rangle(E_k)\right)^{1/p}.$$

Hence, for any non-negative simple functions u and v, we have

$$\int_X \varphi\psi \, d\,|\Gamma\langle f; g\rangle| \leq \left(\int_X \varphi^{p/(p-1)} \, d\Gamma\langle f\rangle\right)^{(p-1)/p} \left(\int_X \psi^p \, d\Gamma\langle g\rangle\right)^{1/p}. \tag{4.15}$$

Next, suppose that u and v are non-negative $\mathcal{B}$-measurable functions and let $\{s_{n,w}\}_{n\geq 1}$ be sequences of non-negative simple functions such that $s_{n,w} \uparrow w$ m-a.e. as $n \to \infty$ for each $w \in \{\varphi, \psi\}$. Then, by (4.15), for any $n \in \mathbb{N}$,

$$\int_X s_{n,u}s_{n,v} \, d\,|\Gamma\langle f; g\rangle| \leq \left(\int_X s_{n,u}^{p/(p-1)} \, d\Gamma\langle f\rangle\right)^{(p-1)/p} \left(\int_X s_{n,v}^p \, d\Gamma\langle g\rangle\right)^{1/p}.$$

It is clear that $\{s_{n,u}s_{n,v}\}_{n\geq 1}$ is a sequence of non-negative simple functions and $s_{n,u}s_{n,v} \uparrow \varphi\psi$ m-a.e. as $n \to \infty$. Hence letting $n \to \infty$ in the inequality above yields (4.13). □

In the following proposition, we show that integrals with respect to p-energy measures satisfying (GC)$_p$ are p-energy forms on (X, m) satisfying (GC)$_p$.

Proposition 4.9. *Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies (GC)$_p$. Then for any $\mathcal{B}$-measurable function $\varphi\colon X \to [0, \infty)$ with $\|\varphi\|_{\sup} < \infty$, $(\int_X \varphi \, d\Gamma\langle\,\cdot\,\rangle, \mathcal{F})$ is a p-energy form on (X, m) satisfying (GC)$_p$.*

Proof. Let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0, p]$, $q_2 \in [p, \infty]$, $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}^{n_1}$ and $T = (T_1, \dots, T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfying (2.1). Similar to (2.20), by using the triangle inequality for the $\ell^{q_2/p}$-norm and the reverse Minkowski inequality (Proposition 2.7) for the $\ell^{q_1/p}$-norm, we see that for any non-negative simple function φ on $(X, \mathcal{B})$,

$$\left\| \left(\left(\int_X \varphi \, d\Gamma\langle T_l(\boldsymbol{u})\rangle\right)^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \leq \left\| \left(\left(\int_X \varphi \, d\Gamma\langle u_k\rangle\right)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \tag{4.16}$$

We can extend (4.16) to any $\mathcal{B}$-measurable function $\varphi\colon X \to [0, \infty]$ by the monotone convergence theorem. The proof is completed. □

The following Fatou type result is useful.

Proposition 4.10. *Assume that $\mathcal{F} \subseteq L^p(X, m)$ and that $\mathcal{F}$ equipped with $\|\cdot\|_{\mathcal{E},1}$ is a Banach space. Let $\varphi\colon X \to [0, \infty)$ be $\mathcal{B}$-measurable and satisfy $\|\varphi\|_{\sup} < \infty$. If $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ converges weakly in $\mathcal{F}$ to $u \in \mathcal{F}$, then*

$$\int_X \varphi \, d\Gamma\langle u\rangle \leq \liminf_{n\to\infty} \int_X \varphi \, d\Gamma\langle u_n\rangle. \tag{4.17}$$

Proof. Let $\{u_{n_k}\}_k$ be a subsequence with $\lim_{k\to\infty}\int_X \varphi\,d\Gamma\langle u_{n_k}\rangle = \liminf_{n\to\infty}\int_X \varphi\,d\Gamma\langle u_n\rangle$. By Mazur's lemma (Lemma 3.13), there exist $N(l)\in\mathbb{N}$ and $\{\alpha_{l,k}\}_{k=l}^{N(l)}\subseteq[0,1]$ such that $N(l)>l$, $\sum_{k=l}^{N(l)}\alpha_{l,k}=1$ and $v_l := \sum_{k=l}^{N(l)}\alpha_{l,k}u_k$ converges to u in $\mathcal{F}$ as $l\to\infty$. We see from the triangle inequality for $\left(\int_X \varphi\,d\Gamma\langle\,\cdot\,\rangle\right)^{1/p}$ that

$$\left(\int_X \varphi\,d\Gamma\langle v_l\rangle\right)^{1/p} \le \sum_{k=l}^{N(l)}\alpha_{l,k}\left(\int_X \varphi\,d\Gamma\langle u_k\rangle\right)^{1/p},$$

which implies (4.17) by letting $l\to\infty$. □

4.2 Extensions of p-energy measures

Let $\mathcal{D}\subseteq\mathcal{F}$ be a linear subspace, which is fixed in the rest of this section. In the rest of this subsection, we assume that there exist p-energy measures $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ dominated by $(\mathcal{E},\mathcal{D})$. We will extend p-energy measures to $\Gamma\langle u\rangle$ for $u\in\overline{\mathcal{D}}^{\mathcal{F}}$ in the following proposition.

Proposition 4.11. *For any $u\in\overline{\mathcal{D}}^{\mathcal{F}}$, there exists a unique measure $\Gamma\langle u\rangle$ on $(X,\mathcal{B})$ such that for any $\{u_n\}_{n\in\mathbb{N}}\subseteq\mathcal{D}$ with $\lim_{n\to\infty}\mathcal{E}(u-u_n)=0$ and any $\mathcal{B}$-measurable function $\varphi\colon X\to[0,\infty)$ with $\|\varphi\|_{\sup}<\infty$,*

$$\int_X \varphi\,d\Gamma\langle u\rangle = \lim_{n\to\infty}\int_X \varphi\,d\Gamma\langle u_n\rangle, \tag{4.18}$$

and $\Gamma\langle u\rangle$ further satisfies $\Gamma\langle u\rangle(X)\le\mathcal{E}(u)$. Moreover, for each such φ, $(\int_X \varphi\,d\Gamma\langle\,\cdot\,\rangle,\overline{\mathcal{D}}^{\mathcal{F}})$ is a p-energy form on (X,m).

Proof. By (EM2)$_p$ and the monotone convergence theorem, for any $\mathcal{B}$-measurable function $\varphi\colon X\to[0,\infty]$ and any $u,v\in\mathcal{D}$,

$$\left(\int_X \varphi\,d\Gamma\langle u+v\rangle\right)^{1/p} \le \left(\int_X \varphi\,d\Gamma\langle u\rangle\right)^{1/p} + \left(\int_X \varphi\,d\Gamma\langle v\rangle\right)^{1/p}. \tag{4.19}$$

In the rest of this proof, let $\varphi\colon X\to[0,\infty)$ be $\mathcal{B}$- measurable and satisfy $\|\varphi\|_{\sup}<\infty$. Let $u\in\overline{\mathcal{D}}^{\mathcal{F}}$ and $\{u_n\}_{n\in\mathbb{N}}\subseteq\mathcal{D}$ satisfy $\lim_{n\to\infty}\mathcal{E}(u-u_n)=0$. By (4.19), $\left\{\int_X \varphi\,d\Gamma\langle u_n\rangle\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $[0,\infty)$ and $\lim_{n\to\infty}\int_X \varphi\,d\Gamma\langle u_n\rangle =: I_u(\varphi)$ is independent of the choice of $\{u_n\}_n$. In addition, we have that

$$\left|\left(\int_X \varphi\,d\Gamma\langle u_n\rangle\right)^{1/p} - I_u(\varphi)^{1/p}\right| \le \|\varphi\|_{\sup}^{1/p}\,\mathcal{E}(u_n-u)^{1/p}, \tag{4.20}$$

that $0\le I_u(\varphi)\le\|\varphi\|_{\sup}\,\mathcal{E}(u)$ and that I_n is linear in the sense that $I_u\left(\sum_{k=1}^N a_k\varphi_k\right) = \sum_{k=1}^N a_k I_u(\varphi_k)$ for any $N\in\mathbb{N}$, $(a_k)_{k=1}^N\subseteq[0,\infty)$ and $\mathcal{B}$-measurable functions $\varphi_k\colon X\to[0,\infty)$ with $\|\varphi_k\|_{\sup}<\infty$, $k\in\{1,\dots,N\}$. Now we define $\Gamma\langle u\rangle(A) := I_u(\mathbb{1}_A)\in[0,\infty)$ for

$A \in \mathcal{B}$, and show that $\Gamma\langle u\rangle$ is a finite measure on $(X, \mathcal{B})$. Clearly, $\Gamma\langle u\rangle$ is finitely additive and $\Gamma\langle u\rangle(X) \leq \mathcal{E}(u) < \infty$. Let us show the countable additivity of $\Gamma\langle u\rangle$. By (4.20), for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $\sup_{A \in \mathcal{B}(X)} \left|\Gamma\langle u\rangle(A)^{1/p} - \Gamma\langle u_n\rangle(A)^{1/p}\right| < \varepsilon$ for any $n \geq N_0$. Let $\{A_k\}_{k\in\mathbb{N}} \subseteq \mathcal{B}$ be a sequence of disjoint measurable sets, and set $B_N := \bigcup_{k=N+1}^{\infty} A_k$ for each $N \in \mathbb{N}$. Then we see that for any $N \in \mathbb{N}$ and any $n \geq N_0$,

$$\left|\Gamma\langle u\rangle\left(\bigcup_{k\in\mathbb{N}} A_k\right) - \sum_{k=1}^{N} \Gamma\langle u\rangle(A_k)\right|^{1/p} = \Gamma\langle u\rangle(B_N)^{1/p} \leq \varepsilon + \Gamma\langle u_n\rangle(B_N)^{1/p},$$

whence $\lim_{N\to\infty} \left|\Gamma\langle u\rangle\left(\bigcup_{k\in\mathbb{N}} A_k\right) - \sum_{k=1}^{N} \Gamma\langle u\rangle(A_k)\right| = 0$, proving the desired countable additivity.

Note that $I_{u+v}(\varphi)^{1/p} \leq I_u(\varphi)^{1/p} + I_v(\varphi)^{1/p}$ for any $u, v \in \overline{\mathcal{D}}^{\mathcal{F}}$ by (4.19) and the definition of $I_\bullet(\varphi)$. This together with the monotone convergence theorem implies the triangle inequality for $\left(\int_X \varphi\, d\Gamma\langle\,\cdot\,\rangle\right)^{1/p}$ on $\overline{\mathcal{D}}^{\mathcal{F}}$; in particular, $(\int_X \varphi\, d\Gamma\langle\,\cdot\,\rangle, \overline{\mathcal{D}}^{\mathcal{F}})$ is a p-energy form on (X, m). Next we show (4.18). Let $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{D}$ be a sequence satisfying $\lim_{n\to\infty} \mathcal{E}(u - u_n) = 0$. By the triangle inequality for $(\int_X \varphi\, d\Gamma\langle\,\cdot\,\rangle, \overline{\mathcal{D}}^{\mathcal{F}})$,

$$\left|\left(\int_X \varphi\, d\Gamma\langle u\rangle\right)^{1/p} - \left(\int_X \varphi\, d\Gamma\langle u_n\rangle\right)^{1/p}\right| \leq \left(\int_X \varphi\, d\Gamma\langle u - u_n\rangle\right)^{1/p} \leq \|\varphi\|_{\sup}^{1/p} \mathcal{E}(u - u_n)^{1/p},$$

which together with (4.20) implies (4.18); indeed,

$$\begin{aligned}
&\left|I_u(\varphi)^{1/p} - \left(\int_X \varphi\, d\Gamma\langle u\rangle\right)^{1/p}\right| \\
&\leq \left|I_u(\varphi)^{1/p} - \left(\int_X \varphi\, d\Gamma\langle u_n\rangle\right)^{1/p}\right| + \left|\left(\int_X \varphi\, d\Gamma\langle u_n\rangle\right)^{1/p} - \left(\int_X \varphi\, d\Gamma\langle u\rangle\right)^{1/p}\right| \\
&\leq 2\, \|\varphi\|_{\sup}^{1/p} \mathcal{E}(u - u_n)^{1/p} \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

□

If in addition $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies $(\mathrm{Cla})_p$, then we can easily see that $\{\Gamma\langle f\rangle\}_{f\in\overline{\mathcal{D}}^{\mathcal{F}}}$ also satisfies $(\mathrm{Cla})_p$. We record this fact in the following proposition.

Proposition 4.12. *Assume that* $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ *satisfies* $(\mathrm{Cla})_p$. *Then* $\{\Gamma\langle f\rangle\}_{f\in\overline{\mathcal{D}}^{\mathcal{F}}}$ *satisfies* $(\mathrm{Cla})_p$.

Proof. It is clear from (4.18) that $\{\Gamma\langle f\rangle\}_{f\in\overline{\mathcal{D}}^{\mathcal{F}}}$ satisfies $(\mathrm{Cla})_p$. □

If $\mathcal{F} \subseteq L^p(X, m)$ and $\mathcal{F}$ equipped with $\|\cdot\|_{\mathcal{E},1}$ is a Banach space, then $(\mathrm{GC})_p$ is also extended to p-energy measures $\{\Gamma\langle f\rangle\}_{f\in\overline{\mathcal{D}}^{\mathcal{F}}}$.

Proposition 4.13. *Assume that* $\mathcal{F} \subseteq L^p(X, m)$, *that* $\mathcal{F}$ *equipped with* $\|\cdot\|_{\mathcal{E},1}$ *is a Banach space and that both* $(\mathcal{E}, \mathcal{D})$ *and* $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ *satisfy* $(\mathrm{GC})_p$. *Then, for any* $\mathcal{B}$*-measurable function* $\varphi\colon X \to [0, \infty)$ *with* $\|\varphi\|_{\sup} < \infty$, $(\int_X \varphi\, d\Gamma\langle\,\cdot\,\rangle, \overline{\mathcal{D}}^{\mathcal{F}})$ *is a* p*-energy form on* (X, m) *satisfying* $(\mathrm{GC})_p$.

Proof. Let us fix $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0, p]$, $q_2 \in [p, \infty]$ and $T = (T_1, \dots, T_{n_2}) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfying (2.1). Let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \bigl(\overline{\mathcal{D}}^{\mathcal{F}}\bigr)^{n_1}$. For each $k \in \{1, \dots, n_1\}$, fix $\{u_{k,n}\}_{n \in \mathbb{N}} \subseteq \mathcal{D}$ so that $\lim_{n\to\infty} \|u_k - u_{k,n}\|_{\mathcal{F},1} = 0$. Set $\boldsymbol{u}_n := (u_{1,n}, \dots, u_{n_1,n})$. By (GC)$_p$ for $(\mathcal{E}, \mathcal{D})$ and (2.1), we know that $\{T_l(\boldsymbol{u}_n)\}_n$ is bounded in $\mathcal{F}$ and that $\lim_{n\to\infty} \|T_l(\boldsymbol{u}_n) - T_l(\boldsymbol{u})\|_{L^p(X,m)} = 0$. Since $\mathcal{F}$ is reflexive (see Proposition 3.12) and $\mathcal{F}$ is continuously embedded in $L^p(X, m)$, we see that $T_l(\boldsymbol{u}) \in \overline{\mathcal{D}}^{\mathcal{F}}$ and that there exists a subsequence $\{T_l(\boldsymbol{u}_{n_j})\}_j$ such that $T_l(\boldsymbol{u}_{n_j})$ weakly converges to $T_l(\boldsymbol{u})$ in $\mathcal{F}$ as $j \to \infty$ for any $l \in \{1, \dots, n_2\}$. If $q_2 < \infty$, then we see from Proposition 4.10 that

$$
\begin{aligned}
\left\| \left(\left(\int_X \varphi \, d\Gamma\langle T_l(\boldsymbol{u}) \rangle \right)^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}}
&\le \left(\sum_{l=1}^{n_2} \liminf_{j\to\infty} \left(\int_X \varphi \, d\Gamma\langle T_l(\boldsymbol{u}_{n_j}) \rangle \right)^{1/p} \right)^{1/q_2} \\
&\le \liminf_{j\to\infty} \left(\sum_{l=1}^{n_2} \left(\int_X \varphi \, d\Gamma\langle T_l(\boldsymbol{u}_{n_j}) \rangle \right)^{1/p} \right)^{1/q_2} \\
&\le \liminf_{j\to\infty} \left(\sum_{k=1}^{n_1} \left(\int_X \varphi \, d\Gamma\langle u_{k,n_j} \rangle \right)^{1/p} \right)^{1/q_1} \\
&= \left\| \left(\left(\int_X \varphi \, d\Gamma\langle u_k \rangle \right)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}.
\end{aligned}
$$

The case $q_2 = \infty$ is similar, so $\bigl(\int_X \varphi \, d\Gamma\langle \cdot \rangle, \overline{\mathcal{D}}^{\mathcal{F}}\bigr)$ satisfies (GC)$_p$. □

4.3 Chain rule and strong locality of p-energy measures

In this subsection, we see that strongly local properties for p-energy measures hold if p-energy measures satisfy a chain rule (see Definition 4.14 below). In addition to the setting specified at the beginning at the previous subsection, we assume that (X, m) satisfies (3.27) and (3.28), that $\mathcal{B} = \mathcal{B}(X)$ and that $\mathcal{D} \subseteq \mathcal{F} \cap C(X)$. We also assume that $\mathcal{F} \subseteq L^p(X, m)$ and equip $\mathcal{F}$ with the norm $\|\cdot\|_{\mathcal{E},1}$.

Definition 4.14 (Chain rules for p-energy measures)**.** (i) We say that $\{\Gamma\langle f \rangle\}_{f \in \mathcal{D}}$ satisfies the chain rule (CL1) if and only if for any $u \in \mathcal{D}$ and any $\Phi \in C^1(\mathbb{R})$, we have $\Phi(u) \in \mathcal{D}$ and

$$
d\Gamma\langle \Phi(u) \rangle = |\Phi'(u)|^p \, d\Gamma\langle u \rangle. \tag{4.21}
$$

(ii) Assume that $\{\Gamma\langle f \rangle\}_{f \in \mathcal{D}}$ satisfies (Cla)$_p$. We say that $\{\Gamma\langle f \rangle\}_{f \in \mathcal{D}}$ satisfies the chain rule (CL2) if and only if for any $n \in \mathbb{N}$, $u \in \mathcal{D}$, $\boldsymbol{v} = (v_1, \dots, v_n) \in \mathcal{D}^n$, $\Phi \in C^1(\mathbb{R})$ and $\Psi \in C^1(\mathbb{R}^n)$, we have $\Phi(u), \Psi(\boldsymbol{v}) \in \mathcal{D}$ and

$$
d\Gamma\langle \Phi(u); \Psi(\boldsymbol{v}) \rangle = \sum_{k=1}^{n} \operatorname{sgn}\bigl(\Phi'(u)\bigr) \, |\Phi'(u)|^{p-1} \, \partial_k \Psi(\boldsymbol{v}) \, d\Gamma\langle u; v_k \rangle. \tag{4.22}
$$

Proposition 4.15. *Assume that* $\{\Gamma\langle f \rangle\}_{f \in \mathcal{D}}$ *satisfies* (Cla)$_p$ *and* (CL2)*.*

(a) $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ *satisfies* (CL1).

(b) (Leibniz rule) *For any* $u,v,w\in\mathcal{D}$, *we have* $vw\in\mathcal{D}$ *and*

$$d\Gamma\langle u;vw\rangle = v\,d\Gamma\langle u;w\rangle + w\,d\Gamma\langle u;v\rangle. \tag{4.23}$$

Proof. The statement (a) is clear. Noting that $vw=\frac{1}{4}\bigl[(v+w)^2-(v-w)^2\bigr]$, we immediately have (4.23) from (CL2). □

We have the following theorem as a consequence of (CL1).

Theorem 4.16 (Image density property)**.** *Assume that* $(\mathcal{E},\mathcal{D})$ *satisfies* (2.3), (2.6) *and* (Cla)$_p$, *that* $(\mathcal{F},\|\cdot\|_{\mathcal{E},1})$ *is a Banach space, and that* $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ *satisfies* (CL1). *Then, for any* $u\in\mathcal{D}$, *the Borel measure* $\Gamma\langle u\rangle\circ u^{-1}$ *on* $\mathbb{R}$ *defined by* $\Gamma\langle u\rangle\circ u^{-1}(A):=\Gamma\langle u\rangle(u^{-1}(A))$, $A\in\mathcal{B}(\mathbb{R})$, *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}$.

Proof. This is proved, on the basis of (4.21), in exactly the same way as [Shi24, Proposition 7.6], which is a simple adaptation of [CF, Theorem 4.3.8], but we present the details because in [Shi24] the underlying topological space X is assumed to be a generalized Sierpiński carpet. It suffices to prove that $\Gamma\langle u\rangle\circ u^{-1}(F)=0$ for any $u\in\mathcal{D}$ and any compact subset F of $\mathbb{R}$ such that $\mathscr{L}^1(F)=0$, where $\mathscr{L}^1$ denotes the 1-dimensional Lebesgue measure on $\mathbb{R}$. Let $\{\varphi_n\}_{n\in\mathbb{N}}\subseteq C_c(\mathbb{R})$ satisfy $|\varphi_n|\le 1$, $\lim_{n\to\infty}\varphi_n(x)=\mathbb{1}_F(x)$ for any $x\in\mathbb{R}$ and

$$\int_0^\infty \varphi_n(t)\,dt=\int_{-\infty}^0\varphi_n(t)\,dt=0 \quad \text{for any } n\in\mathbb{N}.$$

We define $\Phi_n(x):=\int_0^x\varphi_n(t)\,dt$, $x\in\mathbb{R}$, and $u_n:=\Phi_n\circ u$ for any $n\in\mathbb{N}$. Then we easily see that $\Phi_n\in C^1(\mathbb{R})\cap C_c(\mathbb{R})$, $\Phi_n(0)=0$, and $\Phi_n'=\varphi_n$ for any $n\in\mathbb{N}$. Also, u_n converges to 0 in $L^p(X,m)$ as $n\to\infty$ by the dominated convergence theorem. By (2.3) for $(\mathcal{E},\mathcal{D})$, we deduce that $u_n\in\mathcal{F}$ and $\sup_{n\in\mathbb{N}}\mathcal{E}(u_n)<\infty$. Since $\mathcal{F}$ is reflexive by Proposition 3.12 and $\mathcal{F}$ is continuously embedded in $L^p(X,m)$, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ weakly converging to 0 in $\mathcal{F}$. By Mazur's lemma, there exist $N(l)\in\mathbb{N}$ and $\{a_{l,k}\}_{k=l}^{N(l)}\subseteq[0,1]$ such that $N(l)>l$, $\sum_{k=l}^{N(l)}a_{l,k}=1$ and $\sum_{k=l}^{N(l)}a_{l,k}u_{n_k}$ converges to 0 in $\mathcal{F}$ as $l\to\infty$. Let us define $\Psi_l\in C^1(\mathbb{R})$ by $\Psi_l:=\sum_{k=l}^{N(l)}a_{l,k}\Phi_{n_k}$. Then $\Psi_l(0)=0$ and $\lim_{l\to\infty}\Psi_l'(x)=\mathbb{1}_F(x)$ for any $x\in\mathbb{R}$. Furthermore, by Fatou's lemma, (4.21) and (EM1)$_p$,

$$\begin{aligned}\Gamma\langle u\rangle\circ u^{-1}(F)&=\int_{\mathbb{R}}\lim_{l\to\infty}|\Psi_l'(t)|^p\,\bigl(\Gamma\langle u\rangle\circ u^{-1}\bigr)(dt)\\&\le\liminf_{l\to\infty}\int_X|\Psi_l'(u(x))|^p\,\Gamma\langle u\rangle(dx)\\&=\liminf_{l\to\infty}\Gamma\langle\Psi_l(u)\rangle(X)\le\liminf_{l\to\infty}\mathcal{E}\bigl(\Psi_l(u)\bigr)=0,\end{aligned}$$

which completes the proof. □

The following theorem gives arguably the strongest possible forms of the strong locality of p-energy measures.

Theorem 4.17 (Strong locality of energy measures)**.** *Assume that $(\mathcal{E},\mathcal{D})$ satisfies (2.3), (2.6) and (Cla)$_p$, that $(\mathcal{F},\|\cdot\|_{\mathcal{E},1})$ is a Banach space, and that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies (CL1). Let $u,u_1,u_2,v\in\mathcal{D}$, $a,a_1,a_2,b\in\mathbb{R}$ and $A\in\mathcal{B}$.*

(a) *If $A\subseteq u^{-1}(a)$, then $\Gamma\langle u\rangle(A)=0$.*

(b) *If $A\subseteq (u-v)^{-1}(a)$, then $\Gamma\langle u\rangle(A)=\Gamma\langle v\rangle(A)$.*

(c) *If $A\subseteq u_1^{-1}(a_1)\cup u_2^{-1}(a_2)$, then*

$$\Gamma_{\mathcal{E}}\langle u_1+u_2+v\rangle(A)+\Gamma_{\mathcal{E}}\langle v\rangle(A)=\Gamma_{\mathcal{E}}\langle u_1+v\rangle(A)+\Gamma_{\mathcal{E}}\langle u_2+v\rangle(A). \tag{4.24}$$

If in addition $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies (Cla)$_p$, then for any $A\subseteq u_1^{-1}(a_1)\cup u_2^{-1}(a_2)$,

$$\Gamma_{\mathcal{E}}\langle u_1+u_2;v\rangle(A)=\Gamma_{\mathcal{E}}\langle u_1;v\rangle(A)+\Gamma_{\mathcal{E}}\langle u_2;v\rangle(A). \tag{4.25}$$

(d) *If $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies (Cla)$_p$ and $A\subseteq (u_1-u_2)^{-1}(a)\cup v^{-1}(b)$, then*

$$\Gamma_{\mathcal{E}}\langle u_1;v\rangle(A)=\Gamma_{\mathcal{E}}\langle u_2;v\rangle(A)\quad\text{and}\quad\Gamma_{\mathcal{E}}\langle v;u_1\rangle(A)=\Gamma_{\mathcal{E}}\langle v;u_2\rangle(A). \tag{4.26}$$

Proof. (a): This is immediate from Theorem 4.16.

(b): This follows from (a) and the triangle inequality for $\Gamma_{\mathcal{E}}\langle\,\cdot\,\rangle(A)^{1/p}$.

(c): Set $A_i:=A\cap u_i^{-1}(a_i)$, $i\in\{1,2\}$. We see from (b) that

$$\begin{aligned}
&\Gamma_{\mathcal{E}}\langle u_1+u_2+v\rangle(A)+\Gamma_{\mathcal{E}}\langle v\rangle(A)\\
&=\Gamma_{\mathcal{E}}\langle u_2+v\rangle(A_1)+\Gamma_{\mathcal{E}}\langle u_1+v\rangle(A_2)+\Gamma_{\mathcal{E}}\langle v\rangle(A)\\
&=\Gamma_{\mathcal{E}}\langle u_2+v\rangle(A_1)+\Gamma_{\mathcal{E}}\langle u_1+v\rangle(A_2)+\Gamma_{\mathcal{E}}\langle u_1+v\rangle(A_1)+\Gamma_{\mathcal{E}}\langle u_2+v\rangle(A_2)\\
&=\Gamma_{\mathcal{E}}\langle u_1+v\rangle(A)+\Gamma_{\mathcal{E}}\langle u_2+v\rangle(A),
\end{aligned}$$

which proves (4.24). Note that $\Gamma_{\mathcal{E}}\langle u_1+u_2\rangle(A)=\Gamma_{\mathcal{E}}\langle u_1\rangle(A)+\Gamma_{\mathcal{E}}\langle u_2\rangle(A)$ by (4.24) in the case $v=0$. Next assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{D}}$ satisfies (Cla)$_p$. By using this equality and applying (4.24) with v replaced by tv for $t\in(0,\infty)$, we have

$$\begin{aligned}
&\frac{\Gamma_{\mathcal{E}}\langle u_1+u_2+tv\rangle(A)-\Gamma_{\mathcal{E}}\langle u_1+u_2\rangle(A)}{t}+t^{p-1}\Gamma_{\mathcal{E}}\langle v\rangle(A)\\
&=\frac{\Gamma_{\mathcal{E}}\langle u_1+tv\rangle(A)-\Gamma_{\mathcal{E}}\langle u_1\rangle(A)}{t}+\frac{\Gamma_{\mathcal{E}}\langle u_2+tv\rangle(A)-\Gamma_{\mathcal{E}}\langle u_2\rangle(A)}{t},
\end{aligned}$$

which implies (4.25) by letting $t\downarrow 0$.

(d): The proof will be very similar to that of Proposition 3.30-(a). By applying (4.24) with u_2-u_1,tv,u_1 for $t\in(0,\infty)$ in place of u_1,u_2,v, we have

$$\frac{\Gamma_{\mathcal{E}}\langle u_1+tv\rangle(A)-\Gamma_{\mathcal{E}}\langle u_1\rangle(A)}{t}=\frac{\Gamma_{\mathcal{E}}\langle u_2+tv\rangle(A)-\Gamma_{\mathcal{E}}\langle u_2\rangle(A)}{t},$$

which implies the former equality in (4.26) by letting $t\downarrow 0$. This equality in turn with $v,0,u_1-u_2$ in place of u_1,u_2,v yields the latter equality in (4.26) by the linearity of $\Gamma_{\mathcal{E}}\langle v;\,\cdot\,\rangle(A)$. $\square$

5 p-Energy measures associated with self-similar p-energy forms

In this section, we focus on the self-similar case. We will introduce the self-similarity for p-energy forms and construct p-energy measures with respect to self-similar p-energy forms. Some fundamental properties of p-energy measures will be shown.

5.1 Self-similar structure and related notions

We first recall standard notation and terminology on self-similar structures (see [Kig01, Chapter 1] for example). Throughout this section, we fix a compact metrizable space K, a finite set S with $\#S \geq 2$ and a continuous injective map $F_i\colon K \to K$ for each $i \in S$. We set $\mathcal{L} \coloneqq (K, S, \{F_i\}_{i\in S})$.

Definition 5.1. (1) Let $W_0 \coloneqq \{\emptyset\}$, where $\emptyset$ is an element called the *empty word*, let $W_n \coloneqq S^n = \{w_1 \dots w_n \mid w_i \in S \text{ for } i \in \{1,\dots,n\}\}$ for $n \in \mathbb{N}$ and let $W_* \coloneqq \bigcup_{n\in\mathbb{N}\cup\{0\}} W_n$. For $w \in W_*$, the unique $n \in \mathbb{N}\cup\{0\}$ with $w \in W_n$ is denoted by $|w|$ and called the *length of* w. For $w, v \in W_*$, $w = w_1 \dots w_{n_1}$, $v = v_1 \dots v_{n_2}$, we define $wv \in W_*$ by $wv \coloneqq w_1 \dots w_{n_1} v_1 \dots v_{n_2}$ ($w\emptyset \coloneqq w, \emptyset v \coloneqq v$).

(2) We set $\Sigma \coloneqq S^{\mathbb{N}} = \{\omega_1\omega_2\omega_3 \dots \mid \omega_i \in S \text{ for } i \in \mathbb{N}\}$, which is always equipped with the product topology of the discrete topology on S, and define the *shift map* $\sigma\colon \Sigma \to \Sigma$ by $\sigma(\omega_1\omega_2\omega_3\dots) \coloneqq \omega_2\omega_3\omega_4\dots$. For $i \in S$ we define $\sigma_i\colon \Sigma \to \Sigma$ by $\sigma_i(\omega_1\omega_2\omega_3\dots) \coloneqq i\omega_1\omega_2\omega_3\dots$. For $\omega = \omega_1\omega_2\omega_3\dots \in \Sigma$ and $n \in \mathbb{N}\cup\{0\}$, we write $[\omega]_n \coloneqq \omega_1\dots\omega_n \in W_n$.

(3) For $w = w_1 \dots w_n \in W_*$, we set $F_w \coloneqq F_{w_1} \circ \dots \circ F_{w_n}$ ($F_\emptyset \coloneqq \mathrm{id}_K$), $K_w \coloneqq F_w(K)$, $\sigma_w \coloneqq \sigma_{w_1} \circ \dots \circ \sigma_{w_n}$ ($\sigma_\emptyset \coloneqq \mathrm{id}_\Sigma$) and $\Sigma_w \coloneqq \sigma_w(\Sigma)$.

(4) A finite subset Λ of W_* is called a *partition* of Σ if and only if $\Sigma_w \cap \Sigma_v = \emptyset$ for any $w, v \in \Lambda$ with $w \neq v$ and $\Sigma = \bigcup_{w\in\Lambda} \Sigma_w$.

Definition 5.2. $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is called a *self-similar structure* if and only if there exists a continuous surjective map $\chi\colon \Sigma \to K$ such that $F_i \circ \chi = \chi \circ \sigma_i$ for any $i \in S$. Note that such χ, if it exists, is unique and satisfies $\{\chi(\omega)\} = \bigcap_{n\in\mathbb{N}} K_{[\omega]_n}$ for any $\omega \in \Sigma$.

In the following definition, we recall the definition of *post-critically finite self-similar structures* introduced by Kigami in [Kig93], which is mainly dealt with in Subsection 8.3.

Definition 5.3. Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a self-similar structure.

(1) We define the *critical set* $\mathcal{C}_{\mathcal{L}}$ and the *post-critical set* $\mathcal{P}_{\mathcal{L}}$ of $\mathcal{L}$ by

$$\mathcal{C}_{\mathcal{L}} \coloneqq \chi^{-1}\bigl(\textstyle\bigcup_{i,j\in S,\, i\neq j} K_i \cap K_j\bigr) \qquad \text{and} \qquad \mathcal{P}_{\mathcal{L}} \coloneqq \textstyle\bigcup_{n\in\mathbb{N}} \sigma^n(\mathcal{C}_{\mathcal{L}}). \tag{5.1}$$

$\mathcal{L}$ is called *post-critically finite*, or *p.-c.f.* for short, if and only if $\mathcal{P}_{\mathcal{L}}$ is a finite set.

(2) We set $V_0 \coloneqq \chi(\mathcal{P}_{\mathcal{L}})$, $V_n \coloneqq \bigcup_{w\in W_n} F_w(V_0)$ for $n \in \mathbb{N}$ and $V_* \coloneqq \bigcup_{n\in\mathbb{N}\cup\{0\}} V_n$.

The set V_0 should be considered as the *"boundary"* of the self-similar set K; indeed, by [Kig01, Proposition 1.3.5-(2)], we have

$$K_w \cap K_v = F_w(V_0) \cap F_v(V_0) \text{ for any } w, v \in W_* \text{ with } \Sigma_w \cap \Sigma_v = \emptyset. \tag{5.2}$$

According to [Kig01, Lemma 1.3.11], $V_{n-1} \subseteq V_n$ for any $n \in \mathbb{N}$, and V_* is dense in K if $V_0 \neq \emptyset$.

The family of cells $\{K_w\}_{w \in W_*}$ describes the local topology of a self-similar structure. Indeed, $\{K_{n,x}\}_{n \geq 0}$, where $K_{n,x} \coloneqq \bigcup_{w \in W_n; x \in K_w} K_w$, forms a fundamental system of neighborhoods of $x \in K$ [Kig01, Proposition 1.3.6]. Moreover, the proof of [Kig01, Proposition 1.3.6] implies that any metric d on K giving the original topology of K satisfies

$$\lim_{n \to \infty} \max_{w \in W_n} \operatorname{diam}(K_w, d) = 0. \tag{5.3}$$

Let us recall the notion of self-similar measures.

Definition 5.4 (Self-similar measures)**.** Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be a self-similar structure and let $(\theta_i)_{i \in S} \in (0,1)^S$ satisfy $\sum_{i \in S} \theta_i = 1$. A Borel probability measure m on K is said to be a *self-similar measure on $\mathcal{L}$ with weight* $(\theta_i)_{i \in S}$ if and only if the following equality (of Borel measures on K) holds:

$$m = \sum_{i \in S} \theta_i (F_i)_* m. \tag{5.4}$$

Proposition 5.5 ([Kig01, Section 1.4] and [Kig09, Theorem 1.2.7])**.** *Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be a self-similar structure and let $(\theta_i)_{i \in S} \in (0,1)^S$ satisfy $\sum_{i \in S} \theta_i = 1$. Then there exists a self-similar measure m on $\mathcal{L}$ with weight $(\theta_i)_{i \in S}$. If $K \neq \overline{V_0}^K$, then $m(K_w) = \theta_w$ and $m(F_w(\overline{V_0}^K)) = 0$ for any $w \in W_*$, where $\theta_w \coloneqq \theta_{w_1} \cdots \theta_{w_n}$ for $w = w_1 \cdots w_n \in W_*$ ($\theta_\emptyset \coloneqq 1$).*

5.2 Self-similar p-energy forms and p-energy measures

In this subsection, we introduce the self-similarity for p-energy forms on self-similar structures and define the p-energy measures associated with a given self-similar p-energy form. In the rest of this subsection, we fix a self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$, a σ-algebra $\mathcal{B}$ which contains $\mathcal{B}(K)$, a measure m on $\mathcal{B}$ with $m(O) > 0$ for any non-empty open subset O of K, $p \in (1, \infty)$ and a p-energy form $(\mathcal{E}, \mathcal{F})$ on (K, m) with $\mathcal{F} \subseteq L^0(K, m)$. Also, we assume that K is connected.

Definition 5.6 (Self-similar p-energy form)**.** Let $\boldsymbol{\rho} = (\rho_i)_{i \in S} \in (0, \infty)^S$. A p-energy form $(\mathcal{E}, \mathcal{F})$ on (K, m) is said to be *self-similar on* $(\mathcal{L}, m)$ *with weight* $\boldsymbol{\rho}$ if and only if the following hold:

$$\mathcal{F} \cap C(K) = \{ f \in C(K) \mid f \circ F_i \in \mathcal{F} \text{ for any } i \in S \}, \tag{5.5}$$

$$\mathcal{E}(f) = \sum_{i \in S} \rho_i \mathcal{E}(f \circ F_i) \quad \text{for any } u \in \mathcal{F} \cap C(K). \tag{5.6}$$

Note that for any partition Λ of Σ, (5.6) implies

$$\mathcal{E}(f)=\sum_{w\in\Lambda}\rho_w\mathcal{E}(f\circ F_w),\quad u\in\mathcal{F}\cap C(K), \tag{5.7}$$

where $\rho_w\coloneqq\rho_{w_1}\cdots\rho_{w_n}$ for $w=w_1\dots w_n\in W_*$. Indeed, (5.7) follows from an induction with respect to $\max_{w\in\Lambda}|w|$.

In the rest of this subsection, we assume that $(\mathcal{E},\mathcal{F})$ is a self-similar p-energy form on $\mathcal{L}$ with weight $\boldsymbol{\rho}=(\rho_i)_{i\in S}$. We can see that the two-variable version $\mathcal{E}(f;g)$ also has the following self-similarity.

Proposition 5.7. *Assume that $(\mathcal{E},\mathcal{F}\cap C(K))$ satisfies $\mathrm{(Cla)}_p$. Then*

$$\mathcal{E}(f;g)=\sum_{i\in S}\rho_i\mathcal{E}(f\circ F_i;g\circ F_i)\quad\textit{for any } f,g\in\mathcal{F}\cap C(K). \tag{5.8}$$

Proof. For any $f,g\in\mathcal{F}\cap C(K)$ and $t>0$, we have

$$\frac{\mathcal{E}(f+tg)-\mathcal{E}(f)}{t}=\sum_{i\in S}\rho_i\frac{\mathcal{E}\bigl(f\circ F_i+t(g\circ F_i)\bigr)-\mathcal{E}(f\circ F_i)}{t}.$$

Letting $t\downarrow 0$ yields (5.8). □

Next we will see that p-energy measures are naturally introduced by virtue of the self-similarity of $(\mathcal{E},\mathcal{F})$ (see also [Hin05, MS23+]). For $f\in\mathcal{F}\cap C(K)$, we define a finite measure $\mathfrak{m}_{\mathcal{E}}^{(n)}\langle f\rangle$ on $W_n=S^n$ by putting $\mathfrak{m}_{\mathcal{E}}^{(n)}\langle f\rangle(\{w\})\coloneqq\rho_w\mathcal{E}(f\circ F_w)$ for each $w\in W_n$. Then, by (5.7), $\{\mathfrak{m}_{\mathcal{E}}^{(n)}\langle f\rangle\}_{n\ge 0}$ satisfies the consistency condition and hence Kolmogorov's extension theorem yields a measure $\mathfrak{m}_{\mathcal{E}}\langle f\rangle$ on $\Sigma=S^{\mathbb{N}}$ such that $\mathfrak{m}_{\mathcal{E}}\langle f\rangle(\Sigma_w)=\rho_w\mathcal{E}(f\circ F_w)$ for any $w\in W_*$. In particular, $\mathfrak{m}_{\mathcal{E}}\langle f\rangle(\Sigma)=\mathcal{E}(f)$. Basic properties of $\mathfrak{m}_{\mathcal{E}}\langle\,\cdot\,\rangle$ are collected in the following proposition.

Proposition 5.8. (a) *Assume that $(\mathcal{E},\mathcal{F}\cap C(K))$ satisfies $\mathrm{(GC)}_p$. Then, for any $A\in\mathcal{B}(K)$, $(\mathfrak{m}_{\mathcal{E}}\langle\,\cdot\,\rangle(A),\mathcal{F}\cap C(K))$ is a p-energy form on (K,m) satisfying $\mathrm{(GC)}_p$.*

(b) *Assume that $(\mathcal{E},\mathcal{F}\cap C(K))$ satisfies $\mathrm{(Cla)}_p$. Then, for any $A\in\mathcal{B}(K)$, $(\mathfrak{m}_{\mathcal{E}}\langle\,\cdot\,\rangle(A),\mathcal{F}\cap C(K))$ is a p-energy form on (K,m) satisfying $\mathrm{(Cla)}_p$. In particular, for any $f,g\in\mathcal{F}\cap C(K)$, the following limit exists in $\mathbb{R}$:*

$$\mathfrak{m}_{\mathcal{E}}\langle f;g\rangle(A)\coloneqq\frac{1}{p}\left.\frac{d}{dt}\mathfrak{m}_{\mathcal{E}}\langle f+tg\rangle(A)\right|_{t=0}, \tag{5.9}$$

Moreover, $\mathfrak{m}_{\mathcal{E}}\langle f;g\rangle$ is a singed measure on $(\Sigma,\mathcal{B}(\Sigma))$.

Proof. (a): Let $n_1,n_2\in\mathbb{N}$, $q_1\in(0,p]$ and $q_2\in[p,\infty]$. For any $T=(T_1,\dots,T_{n_2})\colon\mathbb{R}^{n_1}\to\mathbb{R}^{n_2}$ satisfying (2.1) and any $\boldsymbol{u}=(u_1,\dots,u_{n_1})\in\bigl(\mathcal{F}\cap C(K)\bigr)^{n_1}$,

$$\left\|\bigl(\mathfrak{m}_{\mathcal{E}}\langle T_l(\boldsymbol{u})\rangle(A)^{1/p}\bigr)_{l=1}^{n_2}\right\|_{\ell^{q_2}}\le\left\|\bigl(\mathfrak{m}_{\mathcal{E}}\langle u_k\rangle(A)^{1/p}\bigr)_{k=1}^{n_1}\right\|_{\ell^{q_1}},\quad A\in\mathcal{B}(K). \tag{5.10}$$

If $A = \Sigma_w$ for some $w \in W_*$, then (5.10) is clearly true by $(\mathrm{GC})_p$ for $(\mathcal{E}, \mathcal{F})$. By a similar argument using the reverse Minkowski inequality on $\ell^{q_1/p}$ and the Minkowski inequality on $\ell^{q_2/p}$ as in (2.20), (5.10) holds on the finitely additive class generated by $\{\Sigma_w\}_{w \in W_*}$. Hence the monotone class theorem implies that (5.10) holds for any $A \in \mathcal{B}(\Sigma)$.

(b): Note that a special case of (5.10) proves $(\mathrm{Cla})_p$ for $(\mathfrak{m}_{\mathcal{E}}\langle\,\cdot\,\rangle(A), \mathcal{F} \cap C(K))$; see also Proposition 2.2-(e), (f). Then the derivative in (5.9) exists by Proposition 3.5 and (5.10). In addition, $\mathfrak{m}_{\mathcal{E}}\langle f; g\rangle$ turns out to be a signed measure on $(\Sigma, \mathcal{B}(\Sigma))$ by Proposition 4.6. (Even when $(\mathcal{E}, \mathcal{F})$ does not satisfy $(\mathrm{GC})_p$, this argument together with the triangle inequality for $\mathcal{E}^{1/p}$ shows (5.10) in the the case $(n_1, n_2, q_1, q_2) = (2, 1, p, p)$ and $T_1(x, y) = x + y$, i.e., the triangle inequality on $\mathcal{F} \cap C(K)$ for $\mathfrak{m}_{\mathcal{E}}\langle\,\cdot\,\rangle(A)^{1/p}$.) □

We now define a finite Borel measure $\Gamma_{\mathcal{E}}\langle f\rangle$ on K by

$$\Gamma_{\mathcal{E}}\langle f\rangle(A) \coloneqq \mathfrak{m}_{\mathcal{E}}\langle f\rangle \circ \chi^{-1}(A) \coloneqq \mathfrak{m}_{\mathcal{E}}\langle f\rangle(\chi^{-1}(A)), \quad A \in \mathcal{B}(K) \tag{5.11}$$

where $\chi\colon \Sigma \to K$ is the same map as in Definition 5.2. The following proposition states basic properties and the self-similarity of $\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f \in \mathcal{F} \cap C(K)}$.

Proposition 5.9. *Let $\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f \in \mathcal{F} \cap C(K)}$ be the measures defined by (5.11).*

(a) *$\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f \in \mathcal{F} \cap C(K)}$ satisfies $\Gamma_{\mathcal{E}}\langle f\rangle(K) = \mathcal{E}(f)$, in particular $(\mathrm{EM1})_p$, and $(\mathrm{EM2})_p$.*

(b) *For any $f \in \mathcal{F} \cap C(K)$, any $w \in W_*$ and any $n \in \mathbb{N} \cup \{0\}$,*

$$\rho_w \mathcal{E}(f \circ F_w) \le \Gamma_{\mathcal{E}}\langle f\rangle(K_w) \le \sum_{v \in W_n; K_v \cap K_w \neq \emptyset} \rho_v \mathcal{E}(f \circ F_v). \tag{5.12}$$

(c) *Assume that $(\mathcal{E}, \mathcal{F} \cap C(K))$ satisfies $(\mathrm{GC})_p$ and let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0, p]$, $q_2 \in [p, \infty]$. Then for any $T = (T_1, \dots, T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfying (2.1), any $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \bigl(\mathcal{F} \cap C(K)\bigr)^{n_1}$ and any Borel measurable function $\varphi\colon K \to [0, \infty]$, we have*

$$\left\| \left(\left(\int_K \varphi \, d\Gamma_{\mathcal{E}}\langle T_l(\boldsymbol{u})\rangle \right)^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \left(\left(\int_K \varphi \, d\Gamma_{\mathcal{E}}\langle u_k\rangle \right)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \tag{5.13}$$

In particular Proposition 2.2 with $(\int_K \varphi \, d\Gamma_{\mathcal{E}}\langle\,\cdot\,\rangle, \mathcal{F} \cap C(K))$ in place of $(\mathcal{E}, \mathcal{F})$ holds provided $\|\varphi\|_{\sup} < \infty$.

(d) *The following equality holds:*

$$\Gamma_{\mathcal{E}}\langle f\rangle = \sum_{i \in S} \rho_i \Gamma_{\mathcal{E}}\langle f \circ F_i\rangle \circ F_i^{-1} \quad \text{for any } f \in \mathcal{F} \cap C(K). \tag{5.14}$$

(e) *Assume that $(\mathcal{E}, \mathcal{F} \cap C(K))$ satisfies $(\mathrm{Cla})_p$. Then $\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f \in \mathcal{F} \cap C(K)}$ also satisfies $(\mathrm{Cla})_p$ and*

$$\Gamma_{\mathcal{E}}\langle f; g\rangle = \sum_{i \in S} \rho_i \Gamma_{\mathcal{E}}\langle f \circ F_i; g \circ F_i\rangle \circ F_i^{-1} \quad \text{for any } f, g \in \mathcal{F} \cap C(K). \tag{5.15}$$

(f) *Assume that $(\mathcal{E}, \mathcal{F} \cap C(K))$ satisfies $(\mathrm{Cla})_p$. Then $\mathfrak{m}_{\mathcal{E}}\langle f; g\rangle \circ \chi^{-1} = \Gamma_{\mathcal{E}}\langle f; g\rangle$ for any $f, g \in \mathcal{F} \cap C(K)$.*

Proof. (a): We easily have $\Gamma_{\mathcal{E}}(K) = \mathfrak{m}_{\mathcal{E}}\langle f\rangle(\chi^{-1}(K)) = \mathfrak{m}_{\mathcal{E}}\langle f\rangle(\Sigma) = \mathcal{E}(f)$. The proof of $(\mathrm{EM2})_p$ will be included in the proof of (c) below.

(b): This statement is the same as [MS23+, Lemma 9.15], which is easily proved by noting that $\Sigma_w \subseteq \chi^{-1}(K_w) \subseteq \bigcup_{v\in W_n; K_v\cap K_w\neq\emptyset}\Sigma_v$.

(c): Assume that $(\mathcal{E},\mathcal{F})$ satisfies $(\mathrm{GC})_p$. Let us fix $T = (T_1,\dots,T_{n_2})\colon \mathbb{R}^{n_1}\to\mathbb{R}^{n_2}$ satisfying (2.1) and $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in \bigl(\mathcal{F}\cap C(K)\bigr)^{n_1}$. Let $B \in \mathcal{B}(K)$. By $(\mathrm{GC})_p$ for $(\mathfrak{m}_{\mathcal{E}}\langle\,\cdot\,\rangle(\chi^{-1}(B)), \mathcal{F}\cap C(K))$ (see Proposition 5.8-(a)), we obtain

$$\left\|\bigl(\Gamma_{\mathcal{E}}\langle T_l(\boldsymbol{u})\rangle(B)^{1/p}\bigr)_{l=1}^{n_2}\right\|_{\ell^{q_2}} \le \left\|\bigl(\Gamma_{\mathcal{E}}\langle u_k\rangle(B)^{1/p}\bigr)_{k=1}^{n_1}\right\|_{\ell^{q_1}}, \quad B\in\mathcal{B}(K). \tag{5.16}$$

Again by a similar argument as in (2.20), we see that (5.13) holds for any non-negative Borel measurable simple function φ on K. We get the desired extension, (5.13) for any Borel measurable function $\varphi\colon K\to[0,\infty]$, by the monotone convergence theorem.

(d): The proof is very similar to [Shi24, Proof of Theorem 7.5]. Let $k\in\mathbb{N}$, $w = w_1\dots w_k \in W_k$ and $n\in\mathbb{N}$. We see that

$$\begin{aligned}\sum_{i\in S}\rho_i\mathfrak{m}_{\mathcal{E}}\langle f\circ F_i\rangle(\sigma_i^{-1}(\Sigma_w)) &= \rho_{w_1}\mathfrak{m}_{\mathcal{E}}\langle f\circ F_{w_1}\rangle(\sigma_{w_1}^{-1}(\Sigma_w)) = \rho_{w_1}\mathfrak{m}_{\mathcal{E}}\langle f\circ F_{w_1}\rangle(\Sigma_{w_2\dots w_k})\\ &= \rho_{w_1}\rho_{w_2\dots w_k}\mathcal{E}((f\circ F_{w_1})\circ F_{w_2\dots w_k}) = \mathfrak{m}_{\mathcal{E}}\langle f\rangle(\Sigma_w)\end{aligned}$$

Since $w\in W_*$ is arbitrary, by Dynkin's π-λ theorem, we deduce that

$$\mathfrak{m}_{\mathcal{E}}\langle f\rangle(A) = \sum_{i\in S}\rho_i\mathfrak{m}_{\mathcal{E}}\langle f\circ F_i\rangle\circ\sigma_i^{-1}(A), \quad A\in\mathcal{B}(\Sigma).$$

We obtain (5.14) by $\chi\circ\sigma_i = F_i\circ\chi$.

(e): Assume that $(\mathcal{E},\mathcal{F})$ satisfies $(\mathrm{Cla})_p$. Then $\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f\in\mathcal{F}\cap C(K)}$ satisfies $(\mathrm{Cla})_p$ by (5.16) (see also Proposition 2.2-(e),(f)). Now we obtain (5.15) by letting $t\downarrow 0$ in

$$\Gamma_{\mathcal{E}}\langle f+tg\rangle(A) = \sum_{i\in S}\rho_i\Gamma_{\mathcal{E}}\langle f\circ F_i + t(g\circ F_i)\rangle\bigl(F_i^{-1}(A)\bigr).$$

(f): This is immediate from (5.11), (4.5) and (5.9). □

We next prove the chain rule (CL2) for $\Gamma_{\mathcal{E}}\langle\,\cdot\,\rangle$. Such a chain rule is also obtained in [BV05], but we provide here a self-contained proof because there are some differences from the framework of [BV05].

Theorem 5.10 (Chain rule). *Assume that $\mathbb{R}\mathbb{1}_K \subseteq \mathcal{E}^{-1}(0)$ and that $(\mathcal{E},\mathcal{F}\cap C(K))$ satisfies (2.3), (2.6) and $(\mathrm{Cla})_p$. Then $\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f\in\mathcal{F}\cap C(K)}$ satisfies (CL2), i.e., for any $n\in\mathbb{N}$, $u\in\mathcal{F}\cap C(K)$, $\boldsymbol{v} = (v_1,\dots,v_n) \in \bigl(\mathcal{F}\cap C(K)\bigr)^n$, $\Phi\in C^1(\mathbb{R})$ and $\Psi\in C^1(\mathbb{R}^n)$, we have $\Phi(u),\Psi(\boldsymbol{v})\in\mathcal{F}\cap C(K)$ and*

$$d\Gamma_{\mathcal{E}}\langle\Phi(u);\Psi(\boldsymbol{v})\rangle = \sum_{k=1}^{n}\mathrm{sgn}\bigl(\Phi'(u)\bigr)\,|\Phi'(u)|^{p-1}\,\partial_k\Psi(\boldsymbol{v})\,d\Gamma_{\mathcal{E}}\langle u;v_k\rangle. \tag{5.17}$$

Proof. We easily obtain $\Phi(u), \Psi(\boldsymbol{v}) \in \mathcal{F}$ by Corollary 2.4-(a) and $\mathbb{R}\mathbb{1}_K \subseteq \mathcal{E}^{-1}(0)$. To show (5.17), we will prove

$$\lim_{l\to\infty} \left|\rho_w \mathcal{E}\bigl(\Phi(u\circ F_w); \Psi(\boldsymbol{v}\circ F_w)\bigr) - \mathcal{S}_l(w)\right| = 0 \quad \text{for any } w \in W_*, \tag{5.18}$$

where $x_0 \in K$ is fixed and

$$\mathcal{S}_l(w) := \sum_{\tau\in W_l} \rho_{w\tau}\mathcal{E}\left(\Phi'(u\circ F_{w\tau}(x_0))\cdot(u\circ F_{w\tau}); \sum_{k=1}^{n} \partial_k\Psi(v\circ F_{w\tau}(x_0))\cdot(v_k\circ F_{w\tau})\right), \quad l \in \mathbb{N}\cup\{0\}.$$

We need some preparations to prove (5.18). Note that, for any $z \in W_*$ and $x \in K$,

$$\begin{aligned}
&\Phi\bigl(u(F_z(x))\bigr) - \Phi\bigl(u(F_z(x_0))\bigr) \\
&= \bigl[u(F_z(x)) - u(F_z(x_0))\bigr]\Biggl(\Phi'\bigl(u(F_z(x_0))\bigr) \\
&\qquad + \int_0^1 \Bigl[\Phi'\Bigl(u(F_z(x_0)) + t\bigl(u(F_z(x)) - u(F_z(x_0))\bigr)\Bigr) - \Phi'\bigl(u(F_z(x_0))\bigr)\Bigr]\,dt\Biggr).
\end{aligned}$$

In particular,

$$\Phi(u\circ F_z) - \widehat{u}_z = \Phi\bigl(u(F_z(x_0))\bigr) - \Phi'\bigl(u(F_z(x_0))\bigr)u(F_z(x_0)) + D_z I_z,$$

where $\widehat{u}_z, D_z, I_z \in C(K)$ are given by

$$\begin{aligned}
\widehat{u}_z(x) &:= \Phi'\bigl(u(F_z(x_0))\bigr)\cdot(u\circ F_z)(x), \\
D_z(x) &:= u(F_z(x)) - u(F_z(x_0)), \\
I_z(x) &:= \int_0^1 \Bigl[\Phi'\bigl(u(F_z(x_0)) + tD_z(x)\bigr) - \Phi'\bigl(u(F_z(x_0))\bigr)\Bigr]\,dt, \quad x \in K.
\end{aligned}$$

Hence we have $\left|\rho_w\mathcal{E}\bigl(\Phi(u\circ F_w); \Psi(v\circ F_w)\bigr) - \mathcal{S}_l(w)\right| \le A_{1,l} + A_{2,l}$, where

$$\begin{aligned}
\widehat{v}_z(x) &:= \sum_{k=1}^{n} \partial_k\Psi\bigl(\boldsymbol{v}(F_z(x_0))\bigr)\cdot(v_k\circ F_z)(x) \quad \text{for } z\in W_*, x\in K, \\
A_{1,l} &:= \sum_{\tau\in W_l} \rho_{w\tau}\left|\mathcal{E}\bigl(\Phi(u\circ F_{w\tau}); \Psi(\boldsymbol{v}\circ F_{w\tau})\bigr) - \mathcal{E}\bigl(\Phi(u\circ F_{w\tau}); \widehat{v}_{w\tau}\bigr)\right|, \\
A_{2,l} &:= \sum_{\tau\in W_l} \rho_{w\tau}\left|\mathcal{E}\bigl(\Phi(u\circ F_{w\tau}); \widehat{v}_{w\tau}\bigr) - \mathcal{E}\bigl(\widehat{u}_{w\tau}; \widehat{v}_{w\tau}\bigr)\right|.
\end{aligned}$$

(Note that $\widehat{u}_z, \widehat{v}_z \in \mathcal{F}$ by (5.5).) Next we show $\lim_{l\to\infty} A_{i,l} = 0$ to obtain (5.18). By Corollary 2.4-(a), $I_z \in \mathcal{F}$ and there exists a constant $C_{u,\Phi} \in (0,\infty)$ depending only on $p, \|u\|_{\sup}, \|\Phi'\|_{\sup,[-2\|u\|_{\sup}, 2\|u\|_{\sup}]}$ such that $\mathcal{E}(I_z) \le C_{u,\Phi}\mathcal{E}(u\circ F_z)$ and $\mathcal{E}\bigl(\Phi(u\circ F_z)\bigr) \le C_{u,\Phi}\mathcal{E}(u\circ F_z)$. Therefore, for any $l \in \mathbb{N}\cup\{0\}$,

$$\sum_{\tau\in W_l} \rho_{w\tau}\mathcal{E}\bigl(\Phi(u\circ F_{w\tau}) - \widehat{u}_w\bigr)$$

$$
\begin{aligned}
&= \sum_{\tau\in W_l} \rho_{w\tau}\mathcal{E}(D_w I_w) \\
&\leq 2^{p-1} \sum_{\tau\in W_l} \rho_{w\tau}\Bigl(\|I_w\|_{\sup}^{p}\,\mathcal{E}(D_w) + \|D_w\|_{\sup}^{p}\,\mathcal{E}(I_w)\Bigr) \\
&\leq 2^{p-1}\Bigl(\max_{\tau'\in W_l}\|I_{w\tau'}\|_{\sup}^{p} + \max_{\tau'\in W_l}\|D_{w\tau'}\|_{\sup}^{p}\Bigr)\sum_{\tau\in W_l}\rho_{w\tau}\Bigl(\mathcal{E}(D_{w\tau}) + C_{u,\Phi}\mathcal{E}(u\circ F_{w\tau})\Bigr) \\
&\leq 2^{p-1}(1+C_{u,\Phi})\mathcal{E}(u)\Bigl(\max_{\tau'\in W_l}\|I_{w\tau'}\|_{\sup}^{p} + \max_{\tau'\in W_l}\|D_{w\tau'}\|_{\sup}^{p}\Bigr).
\end{aligned}
$$

Since u and Φ are uniformly continuous on K, we have from (5.3) that both $\max_{\tau'\in W_l}\|I_{w\tau'}\|_{\sup}$ and $\max_{\tau'\in W_l}\|D_{w\tau'}\|_{\sup}$ converge to 0 as $l\to\infty$, and hence

$$
\lim_{l\to\infty}\sum_{\tau\in W_l}\rho_{w\tau}\mathcal{E}\bigl(\Phi(u\circ F_{w\tau}) - \widehat{u}_{w\tau}\bigr) = 0. \tag{5.19}
$$

Similarly, we can show that

$$
\lim_{l\to\infty}\sum_{\tau\in W_l}\rho_{w\tau}\mathcal{E}\bigl(\Psi(\boldsymbol{v}\circ F_{w\tau}) - \widehat{v}_{w\tau}\bigr) = 0. \tag{5.20}
$$

Then, by (3.11), (3.12) and Hölder's inequality, we have

$$
A_{1,l} \lesssim \mathcal{E}(u\circ F_w)^{(p-1)/p}\left(\sum_{\tau\in W_l}\rho_{w\tau}\mathcal{E}(\Psi(\boldsymbol{v}\circ F_{w\tau}) - \widehat{v}_{w\tau})\right)^{1/p},
$$

and

$$
\begin{aligned}
A_{2,l} &\lesssim \sum_{\tau\in W_l}\rho_{w\tau}\mathcal{E}(u\circ F_{w\tau})^{(p-1-\alpha_p)/p}\mathcal{E}\bigl(\Phi(u\circ F_{w\tau}) - \widehat{u}_{w\tau}\bigr)^{\alpha_p/p}\mathcal{E}\bigl(\widehat{v}_{w\tau}\bigr)^{1/p} \\
&\leq \mathcal{E}(u\circ F_w)^{(p-1-\alpha_p)/p}\left(\sum_{\tau\in W_l}\rho_{w\tau}\mathcal{E}(\Phi(u\circ F_{w\tau}) - \widehat{u}_{w\tau})\right)^{\alpha_p/p}\left(\sum_{\tau\in W_l}\rho_{w\tau}\mathcal{E}\bigl(\widehat{v}_{w\tau}\bigr)\right)^{1/p} \\
&\lesssim \mathcal{E}(u\circ F_w)^{(p-1-\alpha_p)/p}\left(\sum_{\tau\in W_l}\rho_{w\tau}\mathcal{E}(\Phi(u\circ F_{w\tau}) - \widehat{u}_{w\tau})\right)^{\alpha_p/p}\max_{k\in\{1,\dots,n\}}\mathcal{E}(v_k\circ F_w)^{1/p}.
\end{aligned}
$$

Combining these estimates with (5.19) and (5.20), we obtain $\lim_{l\to\infty}A_{i,l} = 0$ and thus (5.18) holds.

By the uniform continuities of Φ', $\partial\Psi_k$ and the fact that $\mathfrak{m}_{\mathcal{E}}\langle f; g\rangle(\Sigma_w) = \rho_w\mathcal{E}(f\circ F_w; g\circ F_w)$ for any $f, g\in\mathcal{F}\cap C(K)$ and $w\in W_*$, we easily observe that

$$
\lim_{l\to\infty}\left|\sum_{k=1}^{n}\int_{\Sigma_w}\operatorname{sgn}\bigl(\Phi'(u\circ\chi)\bigr)\left|\Phi'(u\circ\chi)\right|^{p-1}\partial_k\Psi(\boldsymbol{v}\circ\chi)\,d\mathfrak{m}_{\mathcal{E}}\langle u; v_k\rangle - \mathcal{S}_l(w)\right| = 0.
$$

Hence, by (5.18) and the Dynkin class theorem,

$$d\mathfrak{m}_{\mathcal{E}}\langle\Phi(u);\Psi(\boldsymbol{v})\rangle=\sum_{k=1}^{n}\operatorname{sgn}\bigl(\Phi'(u\circ\chi)\bigr)\,|\Phi'(u\circ\chi)|^{p-1}\,\partial_k\Psi(\boldsymbol{v}\circ\chi)\,d\mathfrak{m}_{\mathcal{E}}\langle u;v_k\rangle. \tag{5.21}$$

Then we obtain the desired equality (5.17) by (5.21) and Proposition 5.9-(f). □

In the case $n=1$, $\Psi=\Phi$ and $v_1=u$ in the theorem above, by noting that the proof of (5.17) does not need $(\mathrm{Cla})_p$ for $(\mathcal{E},\mathcal{F})$, we get the following corollary.

Corollary 5.11. *Assume that $\mathbb{R}\mathbb{1}_K\subseteq\mathcal{E}^{-1}(0)$ and that $(\mathcal{E},\mathcal{F}\cap C(K))$ satisfies (2.3) and (2.6). Then $\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f\in\mathcal{F}\cap C(K)}$ satisfies (CL1), i.e., for any $u\in\mathcal{F}\cap C(K)$ and any $\Phi\in C^1(\mathbb{R})$, we have $\Phi(u)\in\mathcal{F}$ and*

$$d\Gamma_{\mathcal{E}}\langle\Phi(u)\rangle=|\Phi'(u)|^{p}\,d\Gamma_{\mathcal{E}}\langle u\rangle. \tag{5.22}$$

We also have the following representation formula (see also [Cap03, Theorem 4.1]).

Proposition 5.12 (Representation formula)**.** *Assume that $\mathbb{R}\mathbb{1}_K\subseteq\mathcal{E}^{-1}(0)$ and that $(\mathcal{E},\mathcal{F})$ satisfies (2.3), (2.6) and $(\mathrm{Cla})_p$. For any $u,\varphi\in\mathcal{F}\cap C(K)$,*

$$\int_X\varphi\,d\Gamma_{\mathcal{E}}\langle u\rangle=\mathcal{E}(u;u\varphi)-\left(\frac{p-1}{p}\right)^{p-1}\mathcal{E}\bigl(|u|^{\frac{p}{p-1}};\varphi\bigr). \tag{5.23}$$

Proof. Define $\Phi\in C^1(\mathbb{R})$ by $\Phi(x):=|x|^{p/(p-1)}$. Note that $\Phi'(x)=\frac{p}{p-1}\operatorname{sgn}(x)\,|x|^{1/(p-1)}$. By Theorem 5.10, we see that

$$\begin{aligned}
&\mathcal{E}(u;u\varphi)-\left(\frac{p-1}{p}\right)^{p-1}\mathcal{E}(\Phi(u);\varphi)\\
&=\int_K u\,d\Gamma_{\mathcal{E}}\langle u;\varphi\rangle+\int_K\varphi\,d\Gamma_{\mathcal{E}}\langle u\rangle-\left(\frac{p-1}{p}\right)^{p-1}\int_K\operatorname{sgn}\bigl(\Phi'(u)\bigr)\,|\Phi'(u)|^{p-1}\,d\Gamma_{\mathcal{E}}\langle u;\varphi\rangle\\
&=\int_K u\,d\Gamma_{\mathcal{E}}\langle u;\varphi\rangle+\int_K\varphi\,d\Gamma_{\mathcal{E}}\langle u\rangle-\left(\frac{p-1}{p}\right)^{p-1}\left(\frac{p}{p-1}\right)^{p-1}\int_K\operatorname{sgn}(u)\,|u|\,d\Gamma_{\mathcal{E}}\langle u;\varphi\rangle\\
&=\int_K\varphi\,d\Gamma_{\mathcal{E}}\langle u\rangle.
\end{aligned}$$

□

In the following corollaries, we recall useful consequences of the chain rule in Theorem 5.10, which are immediate from Theorems 4.16 and 4.17.

Corollary 5.13. *Assume that $\mathbb{R}\mathbb{1}_K\subseteq\mathcal{E}^{-1}(0)$, that $(\mathcal{E},\mathcal{F}\cap C(K))$ satisfies (2.3), (2.6) and $(\mathrm{Cla})_p$, and that $(\mathcal{F},\|\cdot\|_{\mathcal{E},1})$ is a Banach space. Then, for any $u\in\mathcal{F}\cap C(K)$, the Borel measure $\Gamma_{\mathcal{E}}\langle u\rangle\circ u^{-1}$ on $\mathbb{R}$ defined by $\Gamma_{\mathcal{E}}\langle u\rangle\circ u^{-1}(A):=\Gamma_{\mathcal{E}}\langle u\rangle(u^{-1}(A))$, $A\in\mathcal{B}(\mathbb{R})$, is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

Corollary 5.14. *Assume that $\mathbb{R}\mathbb{1}_K\subseteq\mathcal{E}^{-1}(0)$, that $(\mathcal{E},\mathcal{F}\cap C(K))$ satisfies (2.3), (2.6) and $(\mathrm{Cla})_p$, and that $(\mathcal{F},\|\cdot\|_{\mathcal{E},1})$ is a Banach space. Let $u,u_1,u_2,v\in\mathcal{F}\cap C(K)$, $a,a_1,a_2,b\in\mathbb{R}$ and $A\in\mathcal{B}(K)$.*

(a) *If* $A \subseteq u^{-1}(a)$*, then* $\Gamma_{\mathcal{E}}\langle u\rangle(A) = 0$*.*
(b) *If* $A \subseteq (u-v)^{-1}(a)$*, then* $\Gamma_{\mathcal{E}}\langle u\rangle(A) = \Gamma_{\mathcal{E}}\langle v\rangle(A)$*.*
(c) *If* $A \subseteq u_1^{-1}(a_1) \cup u_2^{-1}(a_2)$*, then*

$$\Gamma_{\mathcal{E}}\langle u_1 + u_2 + v\rangle(A) + \Gamma_{\mathcal{E}}\langle v\rangle(A) = \Gamma_{\mathcal{E}}\langle u_1 + v\rangle(A) + \Gamma_{\mathcal{E}}\langle u_2 + v\rangle(A), \tag{5.24}$$

$$\Gamma_{\mathcal{E}}\langle u_1 + u_2; v\rangle(A) = \Gamma_{\mathcal{E}}\langle u_1; v\rangle(A) + \Gamma_{\mathcal{E}}\langle u_2; v\rangle(A). \tag{5.25}$$

(d) *If* $A \subseteq (u_1 - u_2)^{-1}(a) \cup v^{-1}(b)$*, then*

$$\Gamma_{\mathcal{E}}\langle u_1; v\rangle(A) = \Gamma_{\mathcal{E}}\langle u_2; v\rangle(A) \quad \text{and} \quad \Gamma_{\mathcal{E}}\langle v; u_1\rangle(A) = \Gamma_{\mathcal{E}}\langle v; u_2\rangle(A). \tag{5.26}$$

5.3 Extensions of self-similar p-energy measures

As in the previous subsection, we fix a self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$, a σ-algebra $\mathcal{B}$ which contains $\mathcal{B}(K)$, a measure m on $\mathcal{B}$ with $m(O) > 0$ for any non-empty open subset O of K, $p \in (1,\infty)$ and a self-similar p-energy form $(\mathcal{E}, \mathcal{F})$ on $(\mathcal{L}, m)$ with weight $(\rho_i)_{i\in S} \in (0,\infty)^S$. We always equip $\mathcal{F}$ with $\|\cdot\|_{\mathcal{E},1}$ and assume that K is connected. In this setting, we discuss extensions of self-similar p-energy measures to $\overline{\mathcal{F}\cap C(K)}^{\mathcal{F}}$.

Lemma 5.15. *Assume that $\mathcal{F}$ equipped with $\|\cdot\|_{\mathcal{E},1}$ is a Banach space and that m is a self-similar measure on K. Let $u \in \mathcal{F}$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}\cap C(K)$. If $\{u_n\}_{n\in\mathbb{N}}$ converges in $\mathcal{F}$ to u, then $\{u_n \circ F_w\}_{n\in\mathbb{N}}$ converges in $\mathcal{F}$ to $u\circ F_w$ for any $w \in W_*$. In particular,*

$$u\circ F_w \in \overline{\mathcal{F}\cap C(K)}^{\mathcal{F}} \quad \text{for any } u \in \overline{\mathcal{F}\cap C(K)}^{\mathcal{F}} \text{ and any } w \in W_*. \tag{5.27}$$

$$\mathcal{E}(u) = \sum_{i\in S} \rho_i \mathcal{E}(u\circ F_i) \quad \text{for any } u \in \overline{\mathcal{F}\cap C(K)}^{\mathcal{F}}. \tag{5.28}$$

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ satisfy $\lim_{n\to\infty}\|u - u_n\|_{\mathcal{E},1} = 0$. Then we easily see from the self-similarity of m that $\{u_n\circ F_w\}_{n\in\mathbb{N}}$ converges in $L^p(K,m)$ to $u\circ F_w$ for any $w \in W_*$. Since $\mathcal{E}(u_n\circ F_w - u_k\circ F_w) \le \rho_w^{-1}\mathcal{E}(u_n - u_k)$ for any $n, k \in \mathbb{N}$ by (5.6), $\{u_n\circ F_w\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{F}$. Therefore, it has to converge to $u\circ F_w$ in $\mathcal{F}$, which shows (5.27). By letting $n\to\infty$ in (5.6) for u_n, we obtain (5.28). □

Once one obtains the identity (5.28), in a similar way using Kolmogorov's extension theorem as in the previous subsection, one can define a finite Borel measure $\mathfrak{m}_{\mathcal{E}}\langle u\rangle$ on Σ for each $u \in \overline{\mathcal{F}\cap C(K)}^{\mathcal{F}}$ so that $\mathfrak{m}_{\mathcal{E}}\langle u\rangle(\Sigma_w) = \rho_w\mathcal{E}(u\circ F_w)$ for any $w \in W_*$. The following lemma states the triangle inequality for $\mathfrak{m}_{\mathcal{E}}\langle\,\cdot\,\rangle(A)^{1/p}$ on $\overline{\mathcal{F}\cap C(K)}^{\mathcal{F}}$.

Lemma 5.16. *Assume that $\mathcal{F}$ equipped with $\|\cdot\|_{\mathcal{E},1}$ is a Banach space and that m is a self-similar measure on K. Then for any $u, v \in \overline{\mathcal{F}\cap C(K)}^{\mathcal{F}}$ and any $A \in \mathcal{B}(\Sigma)$,*

$$\mathfrak{m}_{\mathcal{E}}\langle u + v\rangle(A)^{1/p} \le \mathfrak{m}_{\mathcal{E}}\langle u\rangle(A)^{1/p} + \mathfrak{m}_{\mathcal{E}}\langle v\rangle(A)^{1/p}.$$

Proof. One can easily obtain the desired triangle inequality by following the argument in the proof of a special case of Proposition 5.9-(c). □

Now we identify the p-energy measures $\{\Gamma_{\mathcal{E}}\langle u\rangle\}_{u\in\overline{\mathcal{F}\cap C(K)}^{\mathcal{F}}}$ obtained by applying Proposition 4.11 for the measures defined in (5.11) with $\{\mathfrak{m}_{\mathcal{E}}\langle u\rangle\circ\chi^{-1}\}_{u\in\overline{\mathcal{F}\cap C(K)}^{\mathcal{F}}}$.

Proposition 5.17. *Assume that $\mathcal{F}$ equipped with $\|\cdot\|_{\mathcal{E},1}$ is a Banach space and that m is a self-similar measure on K. Then for any $u\in\overline{\mathcal{F}\cap C(K)}^{\mathcal{F}}$ and any $A\in\mathcal{B}(K)$,*

$$\Gamma_{\mathcal{E}}\langle u\rangle(A)=\mathfrak{m}_{\mathcal{E}}\langle u\rangle(\chi^{-1}(A)). \tag{5.29}$$

Proof. The equality (5.29) for $u\in\mathcal{F}\cap C(K)$ is obvious from the definition of $\Gamma_{\mathcal{E}}\langle u\rangle$ in (5.11). Then the desired assertion immediately follows from (4.18), Lemma 5.16 and $\sup_{A\in\mathcal{B}(\Sigma)}\mathfrak{m}_{\mathcal{E}}\langle u\rangle(A)\le\mathcal{E}(u)$. □

We conclude this section by seeing that self-similar p-energy measures can be extended to a localized version of $\mathcal{F}$ in Definition 5.19 below. To this end, we need the following lemma.

Lemma 5.18 (Weak locality of self-similar p-energy measures; [MS23+, Lemma 9.6])**.** *Assume that $\mathcal{F}$ equipped with $\|\cdot\|_{\mathcal{E},1}$ is a Banach space and that m is a self-similar measure on K. Let U be an open subset of K. If $u,v\in\overline{\mathcal{F}\cap C(K)}^{\mathcal{F}}$ satisfy $u=v$ m-a.e. on U, then $\Gamma_{\mathcal{E}}\langle u\rangle(U)=\Gamma_{\mathcal{E}}\langle v\rangle(U)$.*

Proof. The proof is exactly the same as [MS23+, Lemma 9.6], but we recall the details here for the reader's convenience. By the inner regularity of $\Gamma_{\mathcal{E}}\langle u\rangle$ and $\Gamma_{\mathcal{E}}\langle v\rangle$ (see, e.g., [Dud, Theorem 7.1.3]), it suffices to show $\Gamma_{\mathcal{E}}\langle u\rangle(A)=\Gamma_{\mathcal{E}}\langle v\rangle(A)$ for any closed subset A of U. Let d be a metric on K giving the original topology of K. By (5.3), we can choose $\delta\in(0,\operatorname{dist}_d(A,K\setminus U))$ and $N\in\mathbb{N}$ so that $\max_{w\in W_n}\operatorname{diam}(K_w,d)<\delta$ for any $n\ge N$. For $n\in\mathbb{N}$, define $C_n:=\{w\in W_n\mid\Sigma_w\cap\chi^{-1}(A)\neq\emptyset\}$. Since $u\circ F_w=v\circ F_w$ (m-a.e. on K) for any $n\ge N$ and any $w\in C_n$, we have

$$\mathfrak{m}_{\mathcal{E}}\langle u\rangle(\Sigma_{C_n})=\sum_{w\in C_n}\rho_w\mathcal{E}(u\circ F_w)=\sum_{w\in C_n}\rho_w\mathcal{E}(v\circ F_w)=\mathfrak{m}_{\mathcal{E}}\langle v\rangle(\Sigma_{C_n}).$$

Since $\{\Sigma_{C_n}\}_{n\in\mathbb{N}}$ is a decreasing sequence satisfying $\bigcap_{n\in\mathbb{N}}\Sigma_{C_n}=\chi^{-1}(A)$ (see [Hin05, Proof of Lemma 4.1] or [MS23+, Proof of Proposition 9.3]), we obtain $\Gamma_{\mathcal{E}}\langle u\rangle(A)=\Gamma_{\mathcal{E}}\langle v\rangle(A)$ by letting $n\to\infty$ in the equality above. □

Definition 5.19. Let U be a non-empty open subset of K.

(1) We define a linear subspace $\mathcal{F}_{\mathrm{loc}}(U)$ of $L^0(U,m|_U)$ by

$$\mathcal{F}_{\mathrm{loc}}(U):=\left\{f\in L^0(U,m|_U)\;\middle|\;\begin{array}{l}f=f^{\#}\ m\text{-a.e. on } V \text{ for some } f^{\#}\in\mathcal{F}\text{ for}\\ \text{each relatively compact open subset } V\text{ of } U\end{array}\right\}. \tag{5.30}$$

(2) Assume that $\mathcal{F}$ equipped with $\|\cdot\|_{\mathcal{E},1}$ is a Banach space and that m is a self-similar measure on K. In this setting, for each $f\in\mathcal{F}_{\mathrm{loc}}(U)$, we further define a measure $\Gamma_{\mathcal{E}}\langle f\rangle$ on U as follows. We first define $\Gamma_{\mathcal{E}}\langle f\rangle(E):=\Gamma_{\mathcal{E}}\langle f^{\#}\rangle(E)$ for each relatively compact Borel subset E of U, with $A\subseteq U$ and $f^{\#}\in\mathcal{F}$ as in (5.30) chosen so that

$E \subseteq A$; this definition of $\Gamma_{\mathcal{E}}\langle f\rangle(E)$ is independent of a particular choice of such A and $f^{\#}$ by Lemma 5.18. We then define $\Gamma_{\mathcal{E}}\langle f\rangle(E) \coloneqq \lim_{n\to\infty}\Gamma_{\mathcal{E}}\langle f\rangle(E\cap A_n)$ for each $E \in \mathcal{B}|_U$, where $\{A_n\}_{n\in\mathbb{N}}$ is a non-decreasing sequence of relatively compact open subsets of U such that $\bigcup_{n\in\mathbb{N}} A_n = U$; it is clear that this definition of $\Gamma_{\mathcal{E}}\langle f\rangle(E)$ is independent of a particular choice of $\{A_n\}_{n\in\mathbb{N}}$, coincides with the previous one when E is relatively compact in U, and gives a Radon measure on U.

5.4 Self-similar p-energy form as a fixed point

In this subsection, we present a standard method to construct a self-similar p-energy form. The main result of this subsection (Theorem 5.21) is essentially the same as the fixed point theorem in [Kig00, Theorem 1.5], but we present the details to show a useful version of this fixed point theorem where a fixed point is explicitly given as a limit.

As in the previous subsection, we fix a self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$, a σ-algebra $\mathcal{B}$ which contains $\mathcal{B}(K)$, a measure m on $\mathcal{B}$ with $m(O) > 0$ for any non-empty open subset O of K and $p \in (1,\infty)$, and a linear subspace $\mathcal{F}$ of $L^p(K,m)$. We assume that K is connected and that $\mathcal{F}$ satisfies the following property:

$$u \circ F_w \in \mathcal{F} \quad \text{for any } u \in \mathcal{F} \text{ and } w \in W_*.$$

We define

$$\mathfrak{E}_p(\mathcal{F}) \coloneqq \{\mathcal{E}\colon \mathcal{F} \to [0,\infty) \mid (\mathcal{E},\mathcal{F}) \text{ is a } p\text{-energy form on } (K,m)\}.$$

Definition 5.20. Let $\boldsymbol{\rho} = (\rho_i)_{i\in S}$. For $n \in \mathbb{N}\cup\{0\}$, we define $\mathcal{S}_{\boldsymbol{\rho},n}\colon \mathfrak{E}_p(\mathcal{F}) \to \mathfrak{E}_p(\mathcal{F})$ by

$$\mathcal{S}_{\boldsymbol{\rho},n}(E)(u) \coloneqq \sum_{w\in W_n} \rho_w E(u\circ F_w) \quad \text{for } E \in \mathfrak{E}_p(\mathcal{F}) \text{ and } u \in \mathcal{F}. \tag{5.31}$$

(Note that the triangle inequality for $\mathcal{S}_{\boldsymbol{\rho},n}(E)^{1/p}$ can be shown easily.) Set $\mathcal{S}_{\boldsymbol{\rho}} \coloneqq \mathcal{S}_{\boldsymbol{\rho},1}$ for simplicity. Clearly, $\mathcal{S}_{\boldsymbol{\rho},n} = \mathcal{S}_{\boldsymbol{\rho}}^n \coloneqq \underbrace{\mathcal{S}_{\boldsymbol{\rho}}\circ\mathcal{S}_{\boldsymbol{\rho}}\circ\cdots\circ\mathcal{S}_{\boldsymbol{\rho}}}_{n}$.

The desired self-similar p-energy form with weight $\boldsymbol{\rho}$ will be constructed as a non-trivial fixed point of $\mathcal{S}_{\boldsymbol{\rho}}$. The following theorem, which can be regarded as a version of [Kig00, Theorem 1.5] in a specific situation, describes when we can find such a fixed point and how it is obtained.

Theorem 5.21. *Let $\boldsymbol{\rho} = (\rho_i)_{i\in S}$ and let $\mathcal{E}^0 \in \mathfrak{E}_p(\mathcal{F})$. Assume that $\mathcal{F}$ equipped with $\|\cdot\|_{\mathcal{E}^0,1}$ is a separable Banach space and that there exists a constant $C \in [1,\infty)$ such that*

$$C^{-1}\mathcal{E}^0(u) \le \mathcal{S}_{\boldsymbol{\rho},n}(\mathcal{E}^0)(u) \le C\mathcal{E}^0(u) \quad \text{for any } u \in \mathcal{F} \text{ and any } n \in \mathbb{N}\cup\{0\}. \tag{5.32}$$

Then there exists $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $n_k < n_{k+1}$ for any $k \in \mathbb{N}$ such that the following limit exists in $[0,\infty)$ for any $u \in \mathcal{F}$:

$$\mathcal{E}(u) \coloneqq \lim_{k\to\infty}\frac{1}{n_k}\sum_{j=0}^{n_k-1}\mathcal{S}_{\boldsymbol{\rho},j}(\mathcal{E}^0)(u). \tag{5.33}$$

Furthermore, $(\mathcal{E},\mathcal{F})$ is a p-energy form on (K,m) satisfying

$$C^{-1}\mathcal{E}^0(u) \le \mathcal{E}(u) \le C\mathcal{E}^0(u) \quad \text{for any } u \in \mathcal{F} \text{ and any } n \in \mathbb{N}\cup\{0\}, \tag{5.34}$$

where C is the constant in (5.32), and

$$\mathcal{E}(u) = \sum_{w\in W_n} \rho_w \mathcal{E}(u\circ F_w) \quad \text{for any } u \in \mathcal{F} \text{ and any } n \in \mathbb{N}\cup\{0\}. \tag{5.35}$$

Proof. The Set $\mathcal{E}^n := n^{-1}\sum_{j=0}^{n-1}\mathcal{S}_{\boldsymbol{\rho},j}(\mathcal{E}^0)$ for $n \in \mathbb{N}$ for simplicity. Then it is clear that $\mathcal{E}^n \in \mathfrak{E}_p(\mathcal{F})$. Let $\mathscr{C}$ be a countable dense subset of $\mathcal{F}$. Since $\{\mathcal{E}^n(u)\}_{n\in\mathbb{N}}$ is bounded in $[0,\infty)$ for any $u \in \mathcal{F}$ by (5.32), by a standard diagonal procedure, there exists $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $n_k < n_{k+1}$ for any $k \in \mathbb{N}$ such that $\{\mathcal{E}^{n_k}(u')\}_{k\in\mathbb{N}}$ is convergent in $[0,\infty)$ for any $u' \in \mathscr{C}$. Let $u \in \mathcal{F}$, $\varepsilon > 0$ and $u_* \in \mathscr{C}$ satisfy $\mathcal{E}^0(u-u_*)^{1/p} < \varepsilon$. Then for any $k,l \in \mathbb{N}$, by the triangle inequality for $\mathcal{E}^n(\,\cdot\,)^{1/p}$ and (5.32),

$$\begin{aligned}
&\left|\mathcal{E}^{n_k}(u)^{1/p} - \mathcal{E}^{n_l}(u)^{1/p}\right| \\
&\le \left|\mathcal{E}^{n_k}(u)^{1/p} - \mathcal{E}^{n_k}(u_*)^{1/p}\right| + \left|\mathcal{E}^{n_k}(u_*)^{1/p} - \mathcal{E}^{n_l}(u_*)^{1/p}\right| + \left|\mathcal{E}^{n_l}(u)^{1/p} - \mathcal{E}^{n_l}(u_*)^{1/p}\right| \\
&\le 2C^{1/p}\varepsilon + \left|\mathcal{E}^{n_k}(u)^{1/p} - \mathcal{E}^{n_l}(u)^{1/p}\right|,
\end{aligned}$$

whence $\limsup_{k\wedge l\to\infty}\left|\mathcal{E}^{n_k}(u)^{1/p} - \mathcal{E}^{n_l}(u)^{1/p}\right| \le 2C^{1/p}\varepsilon$. Therefore $\{\mathcal{E}^{n_k}(u)\}_{k\in\mathbb{N}}$ is convergent in $[0,\infty)$ for any $u \in \mathcal{F}$, so the limit in (5.33) exists. It is clear that $(\mathcal{E},\mathcal{F})$ is a p-energy form on (K,m) satisfying (5.34).

Let us show (5.35). For any $n \in \mathbb{N}$ and any $u \in \mathcal{F}$, we easily see that

$$\frac{1}{n}\mathcal{E}^0(u) + \mathcal{S}_{\boldsymbol{\rho}}(\mathcal{E}^n)(u) = \frac{1}{n}\mathcal{E}^0(u) + \frac{1}{n}\sum_{l=0}^{n-1}\mathcal{S}_{\boldsymbol{\rho},l+1}(\mathcal{E}^0)(u) = \mathcal{E}^n(u) + \frac{1}{n}\mathcal{S}_{\boldsymbol{\rho},n}(\mathcal{E}^0)(u). \tag{5.36}$$

Since $\lim_{k\to\infty}\mathcal{S}_{\rho}(\mathcal{E}^{n_k})(u) = \mathcal{S}_{\rho}(\mathcal{E})(u)$ and $\lim_{k\to\infty} n_k^{-1}\mathcal{S}_{\rho,n_k}(\mathcal{E}^0)(u) = 0$ by (5.32), we obtain $\mathcal{S}_{\sigma_p}(\mathcal{E}) = \mathcal{E}$ by letting $n \to \infty$ along $\{n_k\}_{k\in\mathbb{N}}$ in (5.36). Hence (5.35) holds. □

By virtue of the explicit representation (5.35), the resulting p-energy form $(\mathcal{E},\mathcal{F})$ inherits some nice properties of $(\mathcal{E}^0,\mathcal{F})$. In the following proposition, we see that (GC)_p and the invariance under good transformations are examples of such properties.

Proposition 5.22. *Assume the same conditions as in Theorem 5.21 and let $\mathcal{E}$ be given by (5.35).*

(a) *If $(\mathcal{E}^0,\mathcal{F})$ satisfies (GC)_p, then $(\mathcal{E},\mathcal{F})$ also satisfies (GC)_p.*

(b) *Let $\mathscr{T}$ be a family of Borel measurable maps from K to K. Assume that $u\circ T \in \mathcal{F}$ and $\mathcal{E}^0(u\circ T) = \mathcal{E}^0(u)$ for any $u \in \mathcal{F}$ and any $T \in \mathscr{T}$. Furthermore, we assume that*

$$F_w^{-1}\circ T\circ F_w \in \mathscr{T} \quad \text{for any } w \in W_*. \tag{5.37}$$

Then $\mathcal{E}(u\circ T) = \mathcal{E}(u)$ for any $u \in \mathcal{F}$ and any $T \in \mathscr{T}$.

Proof. (a): Let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,p]$, $q_2 \in [p,\infty]$ and $T = (T_1,\dots,T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy (2.1). Let $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in \mathcal{F}$. Then $T_l(u_k \circ F_w) = T_l(u_k) \circ F_w \in \mathcal{F}$ for any $k \in \{1,\dots,n_1\}$ and any $w \in W_*$ by $(\mathrm{GC})_p$ for $(\mathcal{E}^0,\mathcal{F})$ and Lemma 5.32. If $q_2 < \infty$, then by a similar estimate as (2.20),

$$\begin{aligned}
\sum_{l=1}^{n_2} \mathcal{S}_\rho(\mathcal{E}^0)\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p} &= \sum_{l=1}^{n_2}\left[\sum_{i\in S}\rho_i\mathcal{E}^0(T_l(\boldsymbol{u})\circ F_i)\right]^{q_2/p} \\
&\le \left(\sum_{i\in S}\rho_i\left[\sum_{l=1}^{n_2}\mathcal{E}^0\bigl(T_l(\boldsymbol{u})\circ F_i\bigr)^{q_2/p}\right]^{p/q_2}\right)^{q_2/p} \qquad (\text{by the triangle ineq. for } \|\cdot\|_{\ell^{q_2/p}}) \\
&\overset{(\mathrm{GC})_p}{\le} \left(\sum_{i\in S}\rho_i\left[\sum_{k=1}^{n_1}\mathcal{E}^0(u_k\circ F_i)^{q_1/p}\right]^{p/q_1}\right)^{q_2/p} \\
&\overset{(2.19)}{\le} \left(\sum_{k=1}^{n_1}\left[\sum_{i\in S}\rho_i\mathcal{E}^0(u_k\circ F_i)\right]^{q_1/p}\right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}} = \left(\sum_{k=1}^{n_1}\mathcal{S}_\rho(\mathcal{E}^0)(u_k)^{q_1/p}\right)^{q_2/q_1},
\end{aligned}$$

whence $\left\|\bigl(\mathcal{S}_\rho(\mathcal{E}^0)(T_l(\boldsymbol{u}))^{1/p}\bigr)_{l=1}^{n_2}\right\|_{\ell^{q_2}} \le \left\|\bigl(\mathcal{S}_\rho(\mathcal{E}^0)(u_k)^{1/p}\bigr)_{k=1}^{n_1}\right\|_{\ell^{q_1}}$. The case $q_2 = \infty$ is similar, so $(\mathcal{S}_\rho(\mathcal{E}^0),\mathcal{F})$ satisfies $(\mathrm{GC})_p$. Similarly, one can easily show that $(\mathcal{S}_{\rho,n}(\mathcal{E}^0),\mathcal{F})$ satisfies $(\mathrm{GC})_p$ for any $n \in \mathbb{N}$. Hence $(\mathrm{GC})_p$ for $(\mathcal{E},\mathcal{F})$ holds by (5.35) and Proposition 2.9-(b).

(b): By (5.35), it suffices to prove $\mathcal{S}_{\boldsymbol{\rho},n}(\mathcal{E}^0)(u\circ T) = \mathcal{S}_{\boldsymbol{\rho},n}(\mathcal{E}^0)(u)$ for any $u \in \mathcal{F}$ and any $T \in \mathscr{T}$. We immediately see that

$$\begin{aligned}
\mathcal{S}_{\boldsymbol{\rho},n}(\mathcal{E}^0)(u\circ T) &= \sum_{w\in W_n}\rho_w\mathcal{E}^0((u\circ T)\circ F_w) \\
&= \sum_{w\in W_n}\rho_w\mathcal{E}^0((u\circ F_w)\circ F_w^{-1}\circ T\circ F_w) \\
&\overset{(5.37)}{=} \sum_{w\in W_n}\rho_w\mathcal{E}^0(u\circ F_w) = \mathcal{S}_{\boldsymbol{\rho},n}(\mathcal{E}^0)(u),
\end{aligned}$$

which completes the proof. □

Also, $(\mathcal{E},\mathcal{F})$ in Theorem 5.21 turns out to be strongly local under a mild condition.

Proposition 5.23. *Assume the same conditions as in Theorem 5.21 and let $\mathcal{E}$ be given by (5.35). If $\{u \in \mathcal{F} \mid \mathcal{E}^0(u) = 0\} = \mathbb{R}\mathbb{1}_K$, then $\{u \in \mathcal{F} \mid \mathcal{E}(u) = 0\} = \mathbb{R}\mathbb{1}_K$ and $(\mathcal{E},\mathcal{F})$ satisfies the strongly local property (SL1).*

Proof. It is immediate from (5.34) that $\{u \in \mathcal{F} \mid \mathcal{E}(u) = 0\} = \mathbb{R}\mathbb{1}_K$. We will show (SL1) for $(\mathcal{E},\mathcal{F})$. Let $u_1, u_2, v \in \mathcal{F}$ and $a_1, a_2 \in \mathbb{R}$. Set $A_i \coloneqq \mathrm{supp}_m[u_i - a_i\mathbb{1}_K]$ for $i \in \{1,2\}$ and assume that $A_1 \cap A_2 = \emptyset$. By (5.3), there exists $n \in \mathbb{N}$ such that $(\bigcup_{w\in W_n[A_1]} K_w) \cap (\bigcup_{w\in W_n[A_2]} K_w) = \emptyset$, where $W_n[A_i] \coloneqq \{w \in W_n \mid K_w \cap A_i \neq \emptyset\}$. Note that $u_i \circ F_w = a_i\mathbb{1}_K$ for $w \in W_n \setminus W_n[A_i]$. This together with $\mathcal{E}(\mathbb{1}_K) = 0$ and (5.35) yields that

$$\mathcal{E}(u_1 + u_2 + v)$$

$$\begin{aligned}
&= \sum_{w \in W_n[A_1]} \rho_w \mathcal{E}(u_1 \circ F_w + v \circ F_w) + \sum_{w \in W_n[A_2]} \rho_w \mathcal{E}(u_2 \circ F_w + v \circ F_w) \\
&\qquad\qquad + \sum_{w \in W_n \setminus (W_n[A_1] \cup W_n[A_2])} \rho_w \mathcal{E}(v \circ F_w) \\
&= \mathcal{E}(u_1 + v) + \mathcal{E}(u_2 + v) - \sum_{w \in W_n \setminus W_n[A_1]} \rho_w \mathcal{E}(v \circ F_w) - \sum_{w \in W_n \setminus W_n[A_2]} \rho_w \mathcal{E}(v \circ F_w) \\
&\qquad\qquad + \sum_{w \in W_n \setminus (W_n[A_1] \cup W_n[A_2])} \rho_w \mathcal{E}(v \circ F_w) \\
&= \mathcal{E}(u_1 + v) + \mathcal{E}(u_2 + v) - \mathcal{E}(v),
\end{aligned}$$

which shows (SL1). □

6 p-Resistance forms and nonlinear potential theory

In this section, we will introduce the notion of p-resistance form as a special class of p-energy forms, and investigate harmonic functions with respect to a p-resistance form. In particular, we prove fundamental results on taking the operation of traces of p-resistance forms, weak comparison principle and Hölder continuity estimates for harmonic functions. We also show the elliptic Harnack inequality for non-negative harmonic functions under some assumptions, and introduce the notion of p-resistance metric with respect to a given p-resistance form.

Throughout this section, we fix $p \in (1, \infty)$, a non-empty set X, a linear subspace $\mathcal{F}$ of $\mathbb{R}^X$ and $\mathcal{E} \colon \mathcal{F} \to [0, \infty)$. (This setting corresponds to choosing m as the counting measure on X in the previous sections.)

6.1 Basics of p-resistance forms

The next definition is a L^p-analogue of the notion of *resistance form*; see [Kig01, Kig03, Kig12] for details on resistance forms.

Definition 6.1 (p-Resistance form)**.** The pair $(\mathcal{E}, \mathcal{F})$ of $\mathcal{F} \subseteq \mathbb{R}^X$ and $\mathcal{E} \colon \mathcal{F} \to [0, \infty)$ is said to be a *p-resistance form* on X if and only if it satisfies the following conditions $(\mathrm{RF1})_p$-$(\mathrm{RF5})_p$:

$(\mathrm{RF1})_p$ $\mathcal{F}$ is a linear subspace of $\mathbb{R}^X$ containing $\mathbb{R}\mathbb{1}_X$ and $\mathcal{E}(\,\cdot\,)^{1/p}$ is a seminorm on $\mathcal{F}$ satisfying $\{u \in \mathcal{F} \mid \mathcal{E}(u) = 0\} = \mathbb{R}\mathbb{1}_X$.

$(\mathrm{RF2})_p$ The quotient normed space $(\mathcal{F}/\mathbb{R}\mathbb{1}_X, \mathcal{E}^{1/p})$ is a Banach space.

$(\mathrm{RF3})_p$ If $x \neq y \in X$, then there exists $u \in \mathcal{F}$ such that $u(x) \neq u(y)$.

$(\mathrm{RF4})_p$ For any $x, y \in X$,

$$R_{\mathcal{E}}(x, y) := R_{(\mathcal{E},\mathcal{F})}(x, y) := \sup\left\{ \frac{|u(x) - u(y)|^p}{\mathcal{E}(u)} \,\middle|\, u \in \mathcal{F} \setminus \mathbb{R}\mathbb{1}_X \right\} < \infty. \qquad (6.1)$$

$(\mathrm{RF5})_p$ $(\mathcal{E},\mathcal{F})$ satisfies $(\mathrm{GC})_p$.

Remark 6.2. (1) The notion of 2-resistance form coincides with the original notion of resistance form (see [Kig01, Definition 2.3.1] for the definition of resistance forms) although the condition $(\mathrm{RF5})_2$ is stronger than (RF5) in [Kig01, Definition 2.3.1]. Indeed, we can obtain $(\mathrm{RF5})_2$ by [Kig12, Theorem 3.14] and the explicit definition of $\mathcal{E}_{L_m}$ in [Kig12, Proposition 3.8].

(2) Let $(\mathcal{E},\mathcal{F})$ be a p-resistance form on a finite set V. Then $\mathcal{F}=\mathbb{R}^V$ by $(\mathrm{RF1})_p$, $(\mathrm{RF3})_p$ and $(\mathrm{RF5})_p$ (see also [Kig12, Proposition 3.2]), so we say simply that $\mathcal{E}$ is a p-resistance form on V if V is a finite set.

Example 6.3. (1) Consider the same setting as in Example 3.10-(1) and suppose that Ω is a bounded domain satisfying the strong local Lipschitz condition (see [AF, Paragraph 4.9]). Then the p-energy form $(\int_{\Omega}|\nabla f|^p\,dx, W^{1,p}(\Omega))$ is a p-resistance form on Ω if and only if $p>D$. Indeed, $(\mathrm{RF1})_p$ and $(\mathrm{RF5})_p$ are clear from the definition (we used the boundedness of Ω to ensure $\mathbb{R}\mathbb{1}_{\Omega}\subseteq L^p(\Omega)$), $(\mathrm{RF2})_p$ and $(\mathrm{RF3})_p$ follow from [AF, Theorem 3.3 and Corollary 3.4] for any $p\in(1,\infty)$. If $p>D$, then we can use the Morrey-type inequality [AF, Lemma 4.28] to verify $(\mathrm{RF4})_p$. Conversely, the supremum in (6.1) is not finite when $p\le D$. To see it, we can assume that $x=0\in\Omega$. Let $\delta\in(0,\infty)$ be small enough so that $\overline{B(0,\delta)}\subseteq\Omega$ and $y\notin\overline{B(0,\delta)}$. For all large $n\in\mathbb{N}$ so that $n^{-1}<\delta$, define $u_n\in C(\Omega)$ by

$$u_n(z):=\left(\frac{\log|z|^{-1}-\log\delta^{-1}}{\log n-\log\delta^{-1}}\right)^{+}\wedge 1,\quad z\in\Omega.$$

Then we easily see that $u_n(0)=1$, $u_n(y)=1$ and $u_n\in W^{1,p}(\Omega)$ with

$$\begin{aligned}\int_{\Omega}|\nabla u_n|^p\,dz &\le \left|\frac{1}{\log(n\delta)}\right|^p\int_{B(0,\delta)\setminus B(0,n^{-1})}|z|^{-p}\,dz=|S_{D-1}|\left|\frac{1}{\log(n\delta)}\right|^p\int_{\frac{1}{n}}^{\delta}r^{-p+D-1}\,dr\\ &=\begin{cases}|S_{D-1}|\,|\log(n\delta)|^{-(p-1)} & \text{if } p=D,\\ \frac{|S_{D-1}|}{D-p}\,|\log(n\delta)|^{-p}\left(\delta^{D-p}-n^{-(D-p)}\right) & \text{if } p<D,\end{cases}\end{aligned}$$

where $|S_{D-1}|$ is the volume of the $(D-1)$-dimensional unit sphere. In particular, $\frac{|u_n(x)-u_n(y)|^p}{\||\nabla u_n|\|_{L^p(\Omega)}^p}\to\infty$ as $n\to\infty$, so $(\mathrm{RF4})_p$ does not hold.

(2) The construction of a regular p-energy form on a compact metric space (K,d) in [Kig23, Theorem 3.21] needs the assumption $p>\dim_{\mathrm{ARC}}(K,d)$, where $\dim_{\mathrm{ARC}}(K,d)$ is the Ahlfors regular conformal dimension of (K,d). (See Definition 8.5-(4) for the definition of $\dim_{\mathrm{ARC}}(K,d)$. The same condition $p>\dim_{\mathrm{ARC}}(K,d)$ is also assumed in [Shi24].) This condition $p>\dim_{\mathrm{ARC}}(K,d)$ plays the same role as $p>D$ in (1) above (see also [CCK24, Theorem 1.1]). In Theorem 8.19, we will see that p-energy forms constructed in [Kig23, Theorem 3.21] are indeed p-resistance forms. We also show that p-energy forms on p.-c.f. self-similar sets in [CGQ22, Theorem 5.1] under the condition (**R**) in [CGQ22, p. 18] are p-resistance forms in Theorem 8.42.

(3) Here we recall typical p-resistance forms on finite sets given in [KS23+, Example 2.2-(1)] because these examples are important to construct self-similar p-resistance forms on p.-c.f. self-similar structures in Subsection 8.3. Let V be a non-empty finite set. Note that in this case $\mathcal{E}$ is a p-resistance form on V if and only if $\mathcal{E}\colon \mathbb{R}^V \to [0,\infty)$ satisfies (RF1)$_p$ and (RF5)$_p$; indeed, (RF3)$_p$ is obvious for $\mathcal{F} = \mathbb{R}^V$, (RF2)$_p$ and (RF4)$_p$ are easily implied by (RF1)$_p$ and $\dim(\mathcal{F}/\mathbb{R}\mathbb{1}_V) < \infty$. Now, consider any functional $\mathcal{E}\colon \mathbb{R}^V \to [0,\infty)$ of the form

$$\mathcal{E}(u) = \frac{1}{2}\sum_{x,y\in V} L_{xy}\,|u(x)-u(y)|^p \tag{6.2}$$

for some $L = (L_{xy})_{x,y\in V} \in [0,\infty)^{V\times V}$ such that $L_{xy} = L_{yx}$ for any $x,y\in V$. It is obvious that $\mathcal{E}$ satisfies (RF1)$_p$ if and only if the graph (V, E_L) is connected, where $E_L \coloneqq \{\{x,y\} \mid x,y\in V,\, x\neq y,\, L_{xy}>0\}$. It is also easy to see that $\mathcal{E}$ satisfies (RF5)$_p$. It thus follows that $\mathcal{E}$ is a p-resistance form on V if and only if (V, E_L) is connected. Note that, while any 2-resistance form on V is of the form (6.2) with $p=2$, *the counterpart of this fact for* $p\neq 2$ *is NOT true* unless $\#V \leq 2$.

In the rest of this section, we assume that $(\mathcal{E},\mathcal{F})$ is a p-resistance form on X. Then the following proposition is immediate from the definition of $R_{\mathcal{E}}$ and Theorem 3.22.

Proposition 6.4. (1) *For any* $u\in\mathcal{F}$ *and any* $x,y\in X$,

$$|u(x)-u(y)|^p \leq R_{\mathcal{E}}(x,y)\mathcal{E}(u). \tag{6.3}$$

(2) $R_{\mathcal{E}}^{1/p}$ *is a metric on* X.

(3) $(\mathcal{F}/\mathbb{R}\mathbb{1}_X, \mathcal{E}^{1/p})$ *is a uniformly convex Banach space, and thus it is reflexive.*

In particular, X can be regarded as a metric space equipped with $R_{\mathcal{E}}^{1/p}$. We equip X with the topology induced from $R_{\mathcal{E}}^{1/p}$. Then we note that $\mathcal{F}\subseteq C(X)$.

We introduce the regularity of p-resistance forms as follows.

Definition 6.5 (Regularity)**.** Assume that X is locally compact. $(\mathcal{E},\mathcal{F})$ is said to be *regular* if and only if $\mathcal{F}\cap C_c(X)$ is dense in $C_c(X)$ with respect to the uniform norm.

The regularity ensures the existence of cut-off functions.

Proposition 6.6. *Assume that* X *is locally compact and that* $(\mathcal{E},\mathcal{F})$ *is regular. For any open subsets* U,V *of* X *with* $\overline{V}^X$ *compact and* $\overline{V}^X\subseteq U$, *there exists* $\psi\in\mathcal{F}\cap C_c(X)$ *such that* $0\leq\psi\leq 1$, $\psi=1$ *on an open neighborhood of* $\overline{V}^X$ *and* $\mathrm{supp}[\psi]\subseteq U$. *In particular,* $\mathcal{F}\cap C_c(X)$ *is a special core.*

Proof. Since X is locally compact, we can pick open subsets Ω_1,Ω_2 of X such that $\overline{V}^X\subseteq\Omega_1\subseteq\Omega_2$, $\overline{\Omega_2}^X\subseteq U$ and $\overline{\Omega_2}^X$ is compact. By Urysohn's lemma, there exists $\psi_0\in C_c(X)$ satisfying $0\leq\psi_0\leq 1$, $\psi_0=1$ on Ω_1, and $\mathrm{supp}[\psi_0]\subseteq U$. Since $(\mathcal{E},\mathcal{F})$ is regular, for any $\varepsilon>0$ there exists $\psi_\varepsilon\in\mathcal{F}\cap C_c(X)$ such that $\|\psi_0-\psi_\varepsilon\|_{\sup}<\varepsilon$. Now define $\psi\coloneqq\bigl[(1-2\varepsilon)^{-1}(\psi_\varepsilon-\varepsilon)^+\bigr]\wedge 1$, then $\psi\in\mathcal{F}$ by (RF1)$_p$ and Proposition 2.2-(b). The other desired properties of ψ are obvious. □

We need the following notation to define traces of a p-resistance form later.

Definition 6.7. Let B be a non-empty subset of X. Define a linear subspace $\mathcal{F}|_B$ of $\mathcal{F}$ by $\mathcal{F}|_B = \{u|_B \mid u \in \mathcal{F}\}$.

The following proposition is useful to discuss boundary conditions on finite sets.

Proposition 6.8. *For any subset B of X with $2 \leq \#B < \infty$, we have $\mathcal{F}|_B = \mathbb{R}^B$.*

Proof. It suffices to show that $\mathbb{1}_x^B \in \mathcal{F}|_B$ for any $x \in B$ by virtue of (RF1)$_p$. Let $x \in B$. For each $y \in B \setminus \{x\}$, by (RF1)$_p$ and (RF2)$_p$, there exists $u_y \in \mathcal{F}$ satisfying $u_y(x) = 1$ and $u_y(y) = 0$. Let $f \coloneqq \sum_{y \in B \setminus \{x\}}(u_y^+ \wedge 1)$ and $g \coloneqq \sum_{y \in B \setminus \{x\}}\bigl((1 - u_y)^+ \wedge 1\bigr)$. Then $f, g \in \mathcal{F}$ by (RF1)$_p$ and (RF5)$_p$. Since $f(x) = \#B - 1$, $f|_{B \setminus \{x\}} \leq \#B - 2$, $g(x) = 0$ and $g|_{B \setminus \{x\}} \geq 1$, the function $h \in \mathcal{F}$ given by

$$h \coloneqq \bigl(f - (\#B - 2)(g^+ \wedge 1)\bigr)^+ \wedge 1$$

satisfies $h|_B = \mathbb{1}_x^B$ and hence $\mathbb{1}_x^B \in \mathcal{F}|_B$. □

The next definition is introduced to deal with Dirichlet-type boundary conditions.

Definition 6.9. For a non-empty subset $B \subseteq X$, define

$$\mathcal{F}^0(B) \coloneqq \{u \in \mathcal{F} \mid u(x) = 0 \text{ for any } x \in X \setminus B\}, \quad B^{\mathcal{F}} \coloneqq \bigcap_{u \in \mathcal{F}^0(X \setminus B)} u^{-1}(0).$$

For basic properties of $B^{\mathcal{F}}$, see [Kig12, Chapters 2, 5 and 6]. Here we only recall the following results, which will be used later.

Proposition 6.10 ([Kig12, Theorems 2.5 and 6.3])**.** *Let B be a non-empty subset of X.*

(a) $\mathbb{C}_{\mathcal{F}} \coloneqq \{B \mid B \subseteq X, B = B^{\mathcal{F}}\}$ *satisfies the axiom of closed sets and it defined a topology on X. Moreover, $\{x\} \in \mathbb{C}_{\mathcal{F}}$ for any $x \in X$.*

(b) *For any $B \subseteq X$ and $x \notin B^{\mathcal{F}}$, there exists $u \in \mathcal{F}$ such that $u \in \mathcal{F}^0(X \setminus B)$, $u(x) = 1$ and $0 \leq u \leq 1$.*

(c) *Assume that X is locally compact and that $(\mathcal{E}, \mathcal{F})$ is regular. Then $B = B^{\mathcal{F}}$ for any closed set B of X.*

Proof. The statements (a) and (b) follow from [Kig12, Theorem 2.4 and Lemma 2.5]. The argument showing (R1) ⇒ (R2) in [Kig12, Proof of Theorem 6.3] proves (c). □

For $B \subseteq X$ and $x \notin B^{\mathcal{F}}$, we define

$$R_{\mathcal{E}}(x, B) \coloneqq R_{(\mathcal{E},\mathcal{F})}(x, B) \coloneqq \sup\left\{ \frac{|u(x)|^p}{\mathcal{E}(u)} \;\middle|\; u \in \mathcal{F}^0(X \setminus B), u(x) \neq 0 \right\} < \infty. \tag{6.4}$$

Note that $R_{\mathcal{E}}(x, \{y\}) = R_{\mathcal{E}}(x, y)$ for $y \in X \setminus \{x\}$ by Proposition 6.10-(a).

6.2 Harmonic functions and traces of p-resistance forms

In this subsection, we consider harmonic functions with respect to p-resistance forms and traces of p-resistance forms to subsets of the original domains.

The following proposition states that the variational and distributional formulations of harmonic functions coincide for p-resistance forms.

Proposition 6.11. *Let $h \in \mathcal{F}$ and $B \subseteq X$. Then the following conditions are equivalent:*

(1) $\mathcal{E}(h) = \inf\{\mathcal{E}(u) \mid u \in \mathcal{F}, u|_B = h|_B\}$.

(2) $\mathcal{E}(h;\varphi) = 0$ *for any* $\varphi \in \mathcal{F}^0(X \setminus B)$.

Proof. Let $\varphi \in \mathcal{F}^0(X \setminus B)$ and set $E(t) := \mathcal{E}(h + t\varphi)$ for $t \in \mathbb{R}$. Then E is differentiable by Proposition 3.5. If $\mathcal{E}(h) = \inf\{\mathcal{E}(u) \mid u \in \mathcal{F}, u|_B = h|_B\}$, then E takes its minimum at $t = 0$. Hence $p\mathcal{E}(h;\varphi) = \frac{d}{dt}E(t)\big|_{t=0} = 0$, which implies $\mathcal{E}(h;\varphi) = 0$ and proves (1) $\Rightarrow$ (2).

Conversely, suppose that $\mathcal{E}(h;\varphi) = 0$ for any $\varphi \in \mathcal{F}^0(X \setminus B)$. Let $v \in \mathcal{F}$ with $v|_B = h|_B$. Then $\mathcal{E}(h) - \mathcal{E}(h;v) = \mathcal{E}(h;h-v) = 0$. By (3.11) and Young's inequality,

$$\mathcal{E}(h) = \mathcal{E}(h;v) \le \mathcal{E}(h)^{(p-1)/p}\mathcal{E}(v)^{1/p} \le \frac{p-1}{p}\mathcal{E}(h) + \frac{1}{p}\mathcal{E}(v),$$

which implies $\mathcal{E}(h) \le \mathcal{E}(v)$. Therefore, $\mathcal{E}(h) = \inf\{\mathcal{E}(u) \mid u \in \mathcal{F}, u|_B = h|_B\}$ and the implication (2) $\Rightarrow$ (1) is proved. □

Definition 6.12 ($\mathcal{E}$-harmonic functions)**.** Let $B \subseteq X$ and $h \in \mathcal{F}$. We say that $h \in \mathcal{F}$ is *$\mathcal{E}$-subharmonic* on $X \setminus B$ if and only if

$$\mathcal{E}(h;\varphi) \le 0 \quad \text{for any } \varphi \in \mathcal{F}^0(X \setminus B) \text{ with } \varphi \ge 0. \tag{6.5}$$

We say that $h \in \mathcal{F}$ is *$\mathcal{E}$-superharmonic* on $X \setminus B$ if and only if $-h$ is $\mathcal{E}$-subharmonic on $X \setminus B$. If h is $\mathcal{E}$-subharmonic and $\mathcal{E}$-superharmonic on $X \setminus B$, i.e., h satisfies either (and hence both) of (1) and (2) in Proposition 6.11, then h is called *$\mathcal{E}$-harmonic* on $X \setminus B$. We set $\mathcal{H}_{\mathcal{E},B} := \{h \in \mathcal{F} \mid h \text{ is } \mathcal{E}\text{-harmonic on } X \setminus B\}$.

$\mathcal{E}$-harmonic functions with given boundary values uniquely exist, and their energies under $\mathcal{E}$ define a new p-resistance form on the boundary set, as follows. This new p-resistance form is called the *trace* of $(\mathcal{E},\mathcal{F})$ on the boundary set.

Theorem 6.13. *Let $B \subseteq X$ be non-empty, and define $\mathcal{E}|_B \colon \mathcal{F}|_B \to [0,\infty)$ by*

$$\mathcal{E}|_B(u) := \inf\{\mathcal{E}(v) \mid v \in \mathcal{F}, v|_B = u\}, \quad u \in \mathcal{F}|_B. \tag{6.6}$$

Then $(\mathcal{E}|_B,\mathcal{F}|_B)$ is a p-resistance form on B and $R_{\mathcal{E}|_B} = R_{\mathcal{E}}|_{B\times B}$. Moreover, for any $u \in \mathcal{F}|_B$ there exists a unique $h^{\mathcal{E}}_B[u] \in \mathcal{F}$ such that $h^{\mathcal{E}}_B[u]\big|_B = u$ and $\mathcal{E}\bigl(h^{\mathcal{E}}_B[u]\bigr) = \mathcal{E}|_B(u)$, so that $h^{\mathcal{E}}_B(\mathcal{F}|_B) = \mathcal{H}_{\mathcal{E},B}$, and

$$h^{\mathcal{E}}_B[au + b\mathbb{1}_B] = ah^{\mathcal{E}}_B[u] + b\mathbb{1}_X \quad \text{for any } u \in \mathcal{F}|_B \text{ and any } a,b \in \mathbb{R}, \tag{6.7}$$

$$\mathcal{E}|_B(u;v) = \mathcal{E}\bigl(h^{\mathcal{E}}_B[u];h^{\mathcal{E}}_B[v]\bigr) \quad \text{for any } u,v \in \mathcal{F}|_B, \tag{6.8}$$

$$\mathcal{E}|_B(f|_B;g|_B) = \mathcal{E}(f;g) \quad \text{for any } f \in \mathcal{H}_{\mathcal{E},B} \text{ and any } g \in \mathcal{F}, \tag{6.9}$$

where $\mathcal{E}|_B(u;v) := \frac{1}{p}\frac{d}{dt}\mathcal{E}|_B(u+tv)\big|_{t=0}$ for $u,v \in \mathcal{F}|_B$ (recall (3.8)).

Remark 6.14. The map $h^{\mathcal{E}}_{B}[\,\cdot\,]$ does *not* satisfy either $h^{\mathcal{E}}_{B}[u+v] \leq h^{\mathcal{E}}_{B}[u] + h^{\mathcal{E}}_{B}[u]$ for any $u, v \in \mathcal{F}|_{B}$ or $h^{\mathcal{E}}_{B}[u+v] \geq h^{\mathcal{E}}_{B}[u] + h^{\mathcal{E}}_{B}[u]$ for any $u, v \in \mathcal{F}|_{B}$ in general, unless $p = 2$ or $\#B \leq 2$.

Proof. We first show the desired existence of $h^{\mathcal{E}}_{B}[u]$ for any $u \in \mathcal{F}|_{B}$. Let us fix $y_{*} \in B$ and let $\alpha \coloneqq \inf\bigl\{\mathcal{E}(v) \bigm| v \in \mathcal{F} \text{ with } v|_{B} = u\bigr\} \in [0,\infty)$. Then there exists $\{v_{n}\}_{n\in\mathbb{N}}$ such that $v_{n} \in \mathcal{F}$, $v_{n}|_{B} = u$ and $\mathcal{E}(v_{n}) \leq \alpha + n^{-1}$ for any $n \in \mathbb{N}$. Note that $\frac{v_{k}+v_{l}}{2} \in \mathcal{F}$ also satisfies $\bigl(\frac{v_{k}+v_{l}}{2}\bigr)\big|_{B} = u$ for any $k, l \in \mathbb{N}$. In the case $p \in (1,2]$, we see that

$$\begin{aligned}\mathcal{E}(v_{k}-v_{l})^{1/(p-1)} &\overset{(2.7)}{\leq} 2\bigl(\mathcal{E}(v_{k}) + \mathcal{E}(v_{l})\bigr)^{1/(p-1)} - \mathcal{E}(v_{k}+v_{l})^{1/(p-1)} \\ &\leq 2\bigl(2\alpha + k^{-1} + l^{-1}\bigr)^{1/(p-1)} - 2^{p/(p-1)}\alpha^{1/(p-1)} \\ &\xrightarrow[k\wedge l\to\infty]{} 2(2\alpha)^{1/(p-1)} - 2^{p/(p-1)}\alpha^{1/(p-1)} = 0. \end{aligned} \tag{6.10}$$

Similarly, in the case $p \in [2,\infty)$, we have

$$\begin{aligned}\mathcal{E}(v_{k}-v_{l}) &\overset{(2.9)}{\leq} 2\bigl(\mathcal{E}(v_{k})^{1/(p-1)} + \mathcal{E}(v_{l})^{1/(p-1)}\bigr)^{p-1} - \mathcal{E}(v_{k}+v_{l}) \\ &\leq 2\bigl((\alpha + k^{-1})^{1/(p-1)} + (\alpha + l^{-1})^{1/(p-1)}\bigr)^{p-1} - 2^{p}\alpha \\ &\xrightarrow[k\wedge l\to\infty]{} 2\bigl(2\alpha^{1/(p-1)}\bigr)^{p-1} - 2^{p}\alpha = 0. \end{aligned} \tag{6.11}$$

Consequently, $\{v_{n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{F}/\mathbb{R}\mathbb{1}_{X}, \mathcal{E}^{1/p})$. By (RF2)$_{p}$, there exists $h \in \mathcal{F}$ such that $h(y_{*}) = u(y_{*})$ and $\lim_{n\to\infty}\mathcal{E}(h - v_{n}) = 0$. For any $y \in B$, by (RF4)$_{p}$,

$$|h(y) - u(y)|^{p} = |h(y) - v_{n}(y)|^{p} = |(h-v_{n})(y) - (h-v_{n})(y_{*})|^{p} \leq R_{\mathcal{E}}(y,y_{*})\mathcal{E}(h-v_{n}) \to 0,$$

and hence $h|_{B} = u$. In particular, h is a minimizer of α. Suppose that $g \in \mathcal{F}$ also satisfies $g|_{B} = u$ and $\mathcal{E}(g) = \alpha$. Then a similar estimate to (6.10) or to (6.11) imply that $\mathcal{E}(h-g) = 0$. Since $h - g \in \mathcal{F}^{0}(X \setminus B)$ and $B \neq \emptyset$, we have $h = g =: h^{\mathcal{E}}_{B}[u]$ by (RF1)$_{p}$. The property (6.7) immediately follows from (RF1)$_{p}$ for $(\mathcal{E},\mathcal{F})$.

Next we prove that $(\mathcal{E}|_{B}, \mathcal{F}|_{B})$ is a p-resistance form on B. It is clear that $\mathcal{E}|_{B}(au) = |a|^{p}\,\mathcal{E}|_{B}(u)$ for any $u \in \mathcal{F}|_{B}$. Let us show the triangle inequality for $\mathcal{E}|_{B}(\,\cdot\,)^{1/p}$, Since $(h^{\mathcal{E}}_{B}[u] + h^{\mathcal{E}}_{B}[v])\big|_{B} = u + v$ for any $u, v \in \mathcal{F}|_{B}$, we see that

$$\begin{aligned}\mathcal{E}|_{B}(u+v)^{1/p} &= \mathcal{E}\bigl(h^{\mathcal{E}}_{B}[u+v]\bigr)^{1/p} \leq \mathcal{E}\bigl(h^{\mathcal{E}}_{B}[u] + h^{\mathcal{E}}_{B}[v]\bigr)^{1/p} \\ &\leq \mathcal{E}\bigl(h^{\mathcal{E}}_{B}[u]\bigr)^{1/p} + \mathcal{E}\bigl(h^{\mathcal{E}}_{B}[v]\bigr)^{1/p} = \mathcal{E}|_{B}(u)^{1/p} + \mathcal{E}|_{B}(v)^{1/p}.\end{aligned}$$

By (6.7), we easily see that $\mathcal{F}|_{B}$ contains $\mathbb{R}\mathbb{1}_{B}$. If $u \in \mathcal{F}|_{B}$ satisfies $\mathcal{E}|_{B}(u) = 0$, then $h^{\mathcal{E}}_{B}[u] \in \mathbb{R}\mathbb{1}_{X}$ and hence $h^{\mathcal{E}}_{B}[u]\big|_{B} = u \in \mathbb{R}\mathbb{1}_{B}$. Thus (RF1)$_{p}$ for $(\mathcal{E}|_{B}, \mathcal{F}|_{B})$ holds. To prove (RF2)$_{p}$ for $(\mathcal{E}|_{B}, \mathcal{F}|_{B})$, let $\{u_{n}\} \subseteq \mathcal{F}|_{B}$ satisfy $\lim_{n\wedge m\to\infty}\mathcal{E}|_{B}(u_{n} - u_{m}) = 0$. Then, by the triangle inequality for $\mathcal{E}|_{B}(\,\cdot\,)^{1/p}$, we easily see that $\{\mathcal{E}|_{B}(u_{n})\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $[0,\infty)$. By (Cla)$_{p}$ for $(\mathcal{E},\mathcal{F})$ and a similar argument to (6.10) (or to (6.11)), we have $\lim_{n\wedge m\to\infty}\mathcal{E}\bigl(h^{\mathcal{E}}_{B}[u_{n}] - h^{\mathcal{E}}_{B}[u_{m}]\bigr) = 0$. Hence there exists $h \in \mathcal{F}$ such that $\lim_{n\to\infty}\mathcal{E}(h - h^{\mathcal{E}}_{B}[u_{n}]) = 0$ by (RF2)$_{p}$ for $(\mathcal{E},\mathcal{F})$. Then $\mathcal{E}|_{B}(h|_{B} - u_{n}) \leq \mathcal{E}(h - h^{\mathcal{E}}_{B}[u_{n}]) \to 0$, which proves the completeness of $\bigl(\mathcal{F}|_B/\mathbb{R}\mathbb{1}_B, \mathcal{E}|_B(\,\cdot\,)^{1/p}\bigr)$. The condition (RF3)$_p$ for $\mathcal{F}|_B$ is clear from that of $\mathcal{F}$. The inequality $R_{\mathcal{E}|_B} \le R_{\mathcal{E}}|_{B\times B}$ (and hence (RF4)$_p$ for $(\mathcal{E}|_B, \mathcal{F}|_B)$) follows from the following estimate:

$$\frac{|u(x)-u(y)|^p}{\mathcal{E}|_B(u)} = \frac{\bigl|h_B^{\mathcal{E}}[u](x)-h_B^{\mathcal{E}}[u](y)\bigr|^p}{\mathcal{E}(h_B^{\mathcal{E}}[u])} \le R_{\mathcal{E}}(x,y) \quad \text{for any } x,y\in B,\ u\in\mathcal{F}|_B.$$

To show the converse inequality $R_{\mathcal{E}|_B} \ge R_{\mathcal{E}}|_{B\times B}$, let $x,y\in B$ and let $u\in\mathcal{F}\setminus\mathbb{R}\mathbb{1}_X$. Suppose that $u(x)\neq u(y)$. Then $u|_B\in\mathcal{F}|_B\setminus\mathbb{R}\mathbb{1}_B$ and $\mathcal{E}(u)\ge\mathcal{E}|_B(u|_B)>0$. Therefore,

$$\frac{|u(x)-u(y)|^p}{\mathcal{E}(u)} \le \frac{|u|_B(x)-u|_B(y)|^p}{\mathcal{E}|_B(u|_B)} \le R_{\mathcal{E}|_B}(x,y).$$

The same estimate is clear if $u(x)=u(y)$, so taking the supremum over $u\in\mathcal{F}\setminus\mathbb{R}\mathbb{1}_X$ yields $R_{\mathcal{E}}(x,y)\le R_{\mathcal{E}|_B}(x,y)$. Lastly, we prove (RF5)$_p$ for $(\mathcal{E}|_B,\mathcal{F}|_B)$. Let $n_1,n_2\in\mathbb{N}$, $q_1\in(0,p]$, $q_2\in[p,\infty]$, $\boldsymbol{u}=(u_1,\dots,u_{n_1})\in\bigl(\mathcal{F}|_B\bigr)^{n_1}$, and suppose that $T=(T_1,\dots,T_{n_2})\colon\mathbb{R}^{n_1}\to\mathbb{R}^{n_2}$ satisfies (2.1). Note that $T_l(\boldsymbol{u})=T_l\bigl(h_B^{\mathcal{E}}[u_1],\dots,h_B^{\mathcal{E}}[u_{n_1}]\bigr)\big|_B\in\mathcal{F}|_B$. Therefore, if $q_2<\infty$, then

$$\begin{aligned}\left(\sum_{l=1}^{n_2}\mathcal{E}|_B\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p}\right)^{1/q_2} &\le \left(\sum_{l=1}^{n_2}\mathcal{E}\Bigl(T_l\bigl(h_B^{\mathcal{E}}[u_1],\dots,h_B^{\mathcal{E}}[u_{n_1}]\bigr)\Bigr)^{q_2/p}\right)^{1/q_2}\\ &\le \left(\sum_{k=1}^{n_1}\mathcal{E}\bigl(h_B^{\mathcal{E}}[u_k]\bigr)^{q_1/p}\right)^{1/q_1} = \left(\sum_{k=1}^{n_1}\mathcal{E}|_B(u_k)^{q_1/p}\right)^{1/q_1}.\end{aligned}$$

The case $q_2=\infty$ is similar, so $(\mathcal{E}|_B,\mathcal{F}|_B)$ satisfies (GC)$_p$.

We conclude the proof by showing (6.8) and (6.9). By Proposition 3.5, we know that

$$\lim_{t\downarrow 0}\frac{\mathcal{E}|_B(u\pm tv)-\mathcal{E}|_B(u)}{\pm t} = \frac{d}{dt}\mathcal{E}|_B(u+tv)\bigg|_{t=0},$$

and

$$\lim_{t\downarrow 0}\frac{\mathcal{E}\bigl(h_B^{\mathcal{E}}[u]\pm th_B^{\mathcal{E}}[v]\bigr)-\mathcal{E}\bigl(h_B^{\mathcal{E}}[u]\bigr)}{\pm t} = p\mathcal{E}\bigl(h_B^{\mathcal{E}}[u];h_B^{\mathcal{E}}[v]\bigr).$$

For any $t>0$, we have

$$\begin{aligned}\frac{\mathcal{E}\bigl(h_B^{\mathcal{E}}[u]-th_B^{\mathcal{E}}[v]\bigr)-\mathcal{E}\bigl(h_B^{\mathcal{E}}[u]\bigr)}{-t} &\le \frac{\mathcal{E}|_B(u-tv)-\mathcal{E}|_B(u)}{-t}\\ &\le \frac{\mathcal{E}|_B(u+tv)-\mathcal{E}|_B(u)}{t} \le \frac{\mathcal{E}\bigl(h_B^{\mathcal{E}}[u]+th_B^{\mathcal{E}}[v]\bigr)-\mathcal{E}\bigl(h_B^{\mathcal{E}}[u]\bigr)}{t},\end{aligned}$$

and hence we obtain (6.8) by letting $t\downarrow 0$. If $f\in\mathcal{H}_{\mathcal{E},B}$, i.e., $h_B^{\mathcal{E}}[f|_B]=f$, then $\mathcal{E}(f;g)=\mathcal{E}(f;h_B^{\mathcal{E}}[g])=\mathcal{E}|_B(f|_B;g|_B)$ since $g-h_B^{\mathcal{E}}[g|_B]\in\mathcal{F}^0(X\setminus B)$ for any $g\in\mathcal{F}$. This proves (6.9). □

The following proposition states a compatibility of the operation taking traces.

Proposition 6.15. *Let A, B be subsets of X such that $\emptyset \neq A \subseteq B$. Then $(\mathcal{E}|_B|_A, \mathcal{F}|_B|_A) = (\mathcal{E}|_A, \mathcal{F}|_A)$ and $h^{\mathcal{E}}_B \circ h^{\mathcal{E}|_B}_A = h^{\mathcal{E}}_A$ for any $u \in \mathcal{F}|_A$. In particular, $h^{\mathcal{E}|_B}_A[u] = h^{\mathcal{E}}_A[u]\big|_B$.*

Proof. Clearly, we have $\mathcal{F}|_B|_A = \mathcal{F}|_A$. For any $u \in \mathcal{F}|_A$, we see that

$$\begin{aligned}
\mathcal{E}|_A(u) = \mathcal{E}\bigl(h^{\mathcal{E}}_A[u]\bigr) &\geq \min\bigl\{\mathcal{E}(v) \bigm| v \in \mathcal{F} \text{ such that } v|_B = h^{\mathcal{E}}_A[u]\big|_B\bigr\} \\
&= \mathcal{E}|_B\bigl(h^{\mathcal{E}}_A[u]\big|_B\bigr) \\
&\geq \min\bigl\{\mathcal{E}|_B(w) \bigm| w \in \mathcal{F}|_B \text{ such that } w|_A = h^{\mathcal{E}}_A[u]\big|_A = u\bigr\} \\
&= \mathcal{E}|_B|_A(u) = \mathcal{E}|_B\bigl(h^{\mathcal{E}|_B}_A[u]\bigr) = \mathcal{E}\Bigl(h^{\mathcal{E}}_B\bigl[h^{\mathcal{E}|_B}_A[u]\bigr]\Bigr) \\
&\geq \min\Bigl\{\mathcal{E}(v) \Bigm| v \in \mathcal{F} \text{ such that } v|_A = \bigl(h^{\mathcal{E}}_B \circ h^{\mathcal{E}|_B}_A\bigr)[u]\big|_A = u\Bigr\} = \mathcal{E}|_A(u),
\end{aligned}$$

which implies $\mathcal{E}|_A(u) = \mathcal{E}|_B|_A(u)$ and $\mathcal{E}\bigl(h^{\mathcal{E}}_A[u]\bigr) = \mathcal{E}\bigl((h^{\mathcal{E}}_B \circ h^{\mathcal{E}|_B}_A)[u]\bigr)$. Since restrictions of both functions $h^{\mathcal{E}}_A[u]$ and $(h^{\mathcal{E}}_B \circ h^{\mathcal{E}|_B}_A)[u]$ to A are u, the uniqueness in Theorem 6.13 implies $h^{\mathcal{E}}_A[u] = \bigl(h^{\mathcal{E}}_B \circ h^{\mathcal{E}|_B}_A\bigr)[u]$. Considering the restriction to B yields $h^{\mathcal{E}|_B}_A[u] = h^{\mathcal{E}}_A[u]\big|_B$. □

The following theorem presents an expression of $(\mathcal{E}, \mathcal{F})$ as the "inductive limit" of its traces $\{\mathcal{E}|_V\}_{V \subseteq X, 1 \leq \#V < \infty}$ to finite subsets, which is a straightforward extension of the counterpart for resistance forms given in [Kaj, Corollary 2.37]. This expression can be applied to get a few useful results on convergences of the seminorm $\mathcal{E}^{1/p}$.

Theorem 6.16. *It holds that*

$$\mathcal{F} = \Biggl\{u \in \mathbb{R}^X \Biggm| \sup_{V \subseteq X; 1 \leq \#V < \infty} \mathcal{E}|_V(u|_V) < \infty\Biggr\}, \tag{6.12}$$

$$\mathcal{E}(u) = \sup_{V \subseteq X; 1 \leq \#V < \infty} \mathcal{E}|_V(u|_V) \quad \text{for any } u \in \mathcal{F}. \tag{6.13}$$

Proof. Let us define $(\mathcal{E}_*, \mathcal{F}_*)$ by

$$\mathcal{E}_*(u) \coloneqq \sup_{V \subseteq X; 1 \leq \#V < \infty} \mathcal{E}|_V(u|_V), \quad u \in \mathbb{R}^X,$$

and $\mathcal{F}_* \coloneqq \{u \in \mathbb{R}^X \mid \mathcal{E}_*(u) < \infty\}$. Then $\mathcal{E}_*^{1/p}$ is clearly a seminorm on $\mathcal{F}_*$ and $\{u \in \mathcal{F}_* \mid \mathcal{E}_*(u) = 0\} = \mathbb{R}\mathbb{1}_X$. We first show that, for any $V \subseteq X$ with $1 \leq \#V < \infty$ and any $u \in \mathbb{R}^V$,

$$h^{\mathcal{E}}_V[u] \in \mathcal{F}_* \quad \text{and} \quad \mathcal{E}|_V(u) = \min\{\mathcal{E}_*(v) \mid v \in \mathcal{F}, v|_V = u\} = \mathcal{E}_*\bigl(h^{\mathcal{E}}_V[u]\bigr), \tag{6.14}$$

both of which are obtained by seeing that, for any $U \subseteq X$ with $1 \leq \#U < \infty$,

$$\mathcal{E}|_U\bigl(h^{\mathcal{E}}_V[u]\big|_U\bigr) \leq \mathcal{E}\bigl(h^{\mathcal{E}}_V[u]\bigr) = \mathcal{E}|_V(u).$$

Indeed, taking the supremum over U, we get $\mathcal{E}_*\bigl(h^{\mathcal{E}}_V[u]\bigr) \leq \mathcal{E}|_V(u)$ and hence (6.14) holds. (The converse $\mathcal{E}|_V(u) \leq \mathcal{E}_*\bigl(h^{\mathcal{E}}_V[u]\bigr)$ is clear from the definition.) We also note that $\mathcal{E}_*$

satisfies (Cla)$_p$ since $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ is a p-resistance form for each $Y \subseteq X$ and $\mathcal{E}|_V(u|_V) \leq \mathcal{E}|_U(u|_U)$ for any $U, V \subseteq X$ with $\emptyset \neq V \subseteq U$ and $u \in \mathbb{R}^U$.

The inclusion $\mathcal{F} \subseteq \mathcal{F}_*$ and $\mathcal{E}_* \leq \mathcal{E}$ (on $\mathcal{F}$) easily follow from the following estimate:

$$\mathcal{E}|_V(u|_V) = \mathcal{E}\bigl(h_V^{\mathcal{E}}[u|_V]\bigr) \leq \mathcal{E}(u) \quad \text{for any } u \in \mathcal{F} \text{ and } V \subseteq X \text{ with } 1 \leq \#V < \infty.$$

To show $\mathcal{F}_* \subseteq \mathcal{F}$ and $\mathcal{E} \leq \mathcal{E}_*$, let $u \in \mathcal{F}_*$, let us choose a subset $V_n \subseteq X$ for each $n \in \mathbb{N}$ such that $1 \leq \#V_n < \infty$ and $\mathcal{E}|_{V_n}(u|_{V_n}) \geq \mathcal{E}_*(u) - n^{-1}$, and set $u_n := h_{V_n}^{\mathcal{E}}[u|_{V_n}]$. Then

$$\mathcal{E}_*(u) - n^{-1} \leq \mathcal{E}|_{V_n}(u|_{V_n}) \overset{(6.14)}{=} \mathcal{E}_*(u_n) \overset{(6.14)}{\leq} \mathcal{E}_*(u),$$

which implies that $\lim_{n\to\infty} \mathcal{E}_*(u_n) = \lim_{n\to\infty} \mathcal{E}(u_n) = \mathcal{E}_*(u)$. Using (Cla)$_p$ for $\mathcal{E}_*$ and $\mathcal{E}_*\bigl(\frac{u+u_n}{2}\bigr) \geq \mathcal{E}_*(u_n)$, we easily obtain $\lim_{n\to\infty} \mathcal{E}_*(u - u_n) = 0$ similarly as (6.10) or (6.11). We next show that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{F}/\mathbb{R}\mathbb{1}_X, \mathcal{E}^{1/p})$. From (Cla)$_p$ for $\mathcal{E}$, $\lim_{n\to\infty} \mathcal{E}(u_n) = \lim_{n\to\infty} \mathcal{E}_*(u_n) = \mathcal{E}_*(u)$ and

$$\mathcal{E}(u_k + u_l) \geq \mathcal{E}\bigl(h_{V_k\cup V_l}^{\mathcal{E}}[(u_k + u_l)|_{V_k\cup V_l}]\bigr) \geq 2^p \mathcal{E}|_{V_k\cup V_l}(u|_{V_k\cup V_l}) \overset{(6.14)}{=} 2^p \mathcal{E}_*(u_{k+l}),$$

we can obtain $\lim_{k\wedge l\to\infty} \mathcal{E}(u_k - u_l) = 0$ similarly as (6.10) or (6.11). Hence, by (RF1)$_p$ for $(\mathcal{E}, \mathcal{F})$, there exists $v \in \mathcal{F}$ such that $\lim_{n\to\infty} \mathcal{E}(v - u_n) = 0$. By $\mathcal{E}_* \leq \mathcal{E}$ on $\mathcal{F}$, we conclude that $\lim_{n\to\infty} \mathcal{E}_*(v - u_n) = 0$, which together with the triangle inequality for $\mathcal{E}_*^{1/p}$ and $\lim_{n\to\infty} \mathcal{E}_*(u - u_n) = 0$ implies that $\mathcal{E}_*(u - v) = 0$ and thus $u - v \in \mathbb{R}\mathbb{1}_X$. In particular, $u = (u - v) + v \in \mathcal{F}_*$ and $\mathcal{E}(u) = \lim_{n\to\infty} \mathcal{E}(u_n) = \mathcal{E}_*(u)$, completing the proof. □

Corollary 6.17. *Let $u \in \mathcal{F}$ and let $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$.*

(a) *Assume that $\lim_{n\to\infty}(u_n(x) - u_n(y)) = u(x) - u(y)$ for any $x, y \in X$. Then $\mathcal{E}(u) \leq \liminf_{n\to\infty} \mathcal{E}(u_n)$.*

(b) *$\lim_{n\to\infty} \mathcal{E}(u - u_n) = 0$ if and only if $\limsup_{n\to\infty} \mathcal{E}(u_n) \leq \mathcal{E}(u)$ and $\lim_{n\to\infty}(u_n(x) - u_n(y)) = u(x) - u(y)$ for any $x, y \in X$.*

Proof. Suppose that $u, u_n \in \mathcal{F}$, $n \in \mathbb{N}$, satisfy $\lim_{n\to\infty}(u_n(x) - u_n(y)) = u(x) - u(y)$ for any $x, y \in X$. For any $\varepsilon > 0$, by Theorem 6.16, there exists $V \subseteq X$ with $1 \leq \#V < \infty$ such that $\mathcal{E}|_V(u|_V) > \mathcal{E}(u) - \varepsilon$. Then we have

$$\lim_{n\to\infty} \mathcal{E}|_V(u_n|_V) = \mathcal{E}|_V(u|_V) > \mathcal{E}(u) - \varepsilon,$$

since $\mathbb{R}^V$ is a finite-dimensional vector space, $\mathcal{E}|_V(\,\cdot\,)^{1/p}$ is a seminorm on $\mathbb{R}^V$ and $\lim_{n\to\infty} \max_{x,y\in V} |(u_n(x) - u_n(y)) - (u(x) - u(y))| = 0$. In particular, there exists $N_1 \in \mathbb{N}$ (depending on ε) such that $\mathcal{E}(u_n) \geq \mathcal{E}|_V(u_n|_V) > \mathcal{E}(u) - \varepsilon$ for any $n \geq N_1$ and hence $\liminf_{n\to\infty} \mathcal{E}(u_n) \geq \mathcal{E}(u)$, proving (a). Next, in addition, we assume that $\limsup_{n\to\infty} \mathcal{E}(u_n) \leq \mathcal{E}(u)$. Then $\lim_{n\to\infty} \mathcal{E}(u_n) = \mathcal{E}(u)$. Since $\{\frac{u+u_n}{2}\}_{n\in\mathbb{N}}$ satisfies the same conditions as $\{u_n\}_{n\in\mathbb{N}}$, we obtain $\lim_{n\to\infty} \mathcal{E}\bigl(\frac{u+u_n}{2}\bigr) = \mathcal{E}(u)$. Similar to (6.10) or (6.11), we have from (Cla)$_p$ for $\mathcal{E}$ that $\lim_{n\to\infty} \mathcal{E}(u - u_n) = 0$. The converse part of (b) is clear from (6.3). □

Corollary 6.18. (a) *Let $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq C(\mathbb{R})$ satisfy $\lim_{n\to\infty}\varphi_n(t) = t$ and $|\varphi_n(t) - \varphi_n(s)| \leq |t-s|$ for any $n \in \mathbb{N}$, $s,t \in \mathbb{R}$. Then $\{\varphi_n(u)\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ and $\lim_{n\to\infty}\mathcal{E}(u - \varphi_n(u)) = 0$ for any $u \in \mathcal{F}$.*

(b) *Let $u \in \mathcal{F}$, $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ and $\varphi \in C(\mathbb{R})$ satisfy $\lim_{n\to\infty}\mathcal{E}(u-u_n) = 0$, $\lim_{n\to\infty} u_n(x) = u(x)$ for some $x \in X$, $|\varphi(t) - \varphi(s)| \leq |t-s|$ for any $s,t \in \mathbb{R}$ and $\varphi(u) = u$. Then $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ and $\lim_{n\to\infty}\mathcal{E}(u - \varphi(u_n)) = 0$.*

Proof. The statement (a) is immediate from Corollary 6.17 and (RF5)$_p$, so we show (b). Since, under the assumptions of (b), for any $y \in X$,

$$|u(y) - u_n(y)| \leq R_{\mathcal{E}}(x,y)^{1/p}\mathcal{E}(u-u_n)^{1/p} + |u(x) - u_n(x)| \xrightarrow[n\to\infty]{} 0,$$

we get $\lim_{n\to\infty}\varphi(u_n(y)) = u(y)$. By (RF5)$_p$, we have $\varphi(u_n) \in \mathcal{F}$ and $\limsup_{n\to\infty}\mathcal{E}(\varphi(u_n)) \leq \lim_{n\to\infty}\mathcal{E}(u_n) = \mathcal{E}(u)$, so Corollary 6.17 yields $\lim_{n\to\infty}\mathcal{E}(u - \varphi(u_n)) = 0$. □

If X is separable, then we have the following useful version of Theorem 6.16.

Proposition 6.19. *Assume that X (equipped with the topology induced by $R_{\mathcal{E}}^{1/p}$) is separable. Let $\{\mathcal{V}_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a increasing sequence of finite subsets of X with $\overline{\mathcal{V}_*}^X = X$, where $\mathcal{V}_* \coloneqq \bigcup_{n\in\mathbb{N}\cup\{0\}}\mathcal{V}_n$. We define $(\mathcal{E}', \mathcal{F}')$ by*

$$\mathcal{F}' \coloneqq \left\{u \in C(X) \;\middle|\; \lim_{n\to\infty}\mathcal{E}|_{\mathcal{V}_n}(u|_{\mathcal{V}_n}) < \infty\right\}, \tag{6.15}$$

$$\mathcal{E}'(u) \coloneqq \lim_{n\to\infty}\mathcal{E}|_{\mathcal{V}_n}(u|_{\mathcal{V}_n}) \in [0,\infty), \quad u \in \mathcal{F}'; \tag{6.16}$$

note that $\{\mathcal{E}|_{\mathcal{V}_n}(u|_{\mathcal{V}_n})\}_{n\in\mathbb{N}\cup\{0\}}$ is non-decreasing since $\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$. Then $(\mathcal{E}', \mathcal{F}') = (\mathcal{E}, \mathcal{F})$. Moreover,

$$\lim_{n\to\infty}\mathcal{E}\bigl(u - h_{\mathcal{V}_n}^{\mathcal{E}}[u|_{\mathcal{V}_n}]\bigr) = 0 \quad \text{for any } u \in \mathcal{F}, \quad \text{and} \tag{6.17}$$

$$\mathcal{E}(u;v) = \lim_{n\to\infty}\mathcal{E}|_{\mathcal{V}_n}(u|_{\mathcal{V}_n}; v|_{\mathcal{V}_n}) \quad \text{for any } u,v \in \mathcal{F}. \tag{6.18}$$

Proof. By Theorem 6.16, $\mathcal{E}' \leq \mathcal{E}$ and $\mathcal{F} \subseteq \mathcal{F}'$ are clear. To show the converse, let $u \in \mathcal{F}'$, set $u_n \coloneqq h_{\mathcal{V}_n}^{\mathcal{E}}(u|_{\mathcal{V}_n}) \in \mathcal{F}$ and fix $x_0 \in \mathcal{V}_0$. We can assume that $u(x_0) = 0$ by considering $u - u(x_0)$ instead of u. A similar estimate to (6.10) or (6.11) for $\mathcal{E}$ and (RF2)$_p$ together imply that $\lim_{n\to\infty}\mathcal{E}(v - u_n) = 0$ for some $v \in \mathcal{F}$ with $v(x_0) = 0$. Since $|v(x) - u(x)|^p \leq R_{\mathcal{E}}(x,x_0)\mathcal{E}(u-u_n)$ for any $x \in \mathcal{V}_*$ and any $n \in \mathbb{N}$ with $x \in \mathcal{V}_n$ by (6.3), we have $v|_{\mathcal{V}_*} = u|_{\mathcal{V}_*}$. By $\overline{\mathcal{V}_*}^X = X$ and $u, v \in C(X)$ (see (6.3)), we conclude that $u = v \in \mathcal{F}$ and thus $\mathcal{F} = \mathcal{F}'$, $\mathcal{E}(u) = \mathcal{E}'(u)$ and $\lim_{n\to\infty}\mathcal{E}(u - u_n) = 0$, i.e., $(\mathcal{E}', \mathcal{F}') = (\mathcal{E}, \mathcal{F})$ and (6.17) hold. The convergence in (6.18) is immediate from (6.17), (3.11) and (3.12). □

Based on Proposition 7.4, a standard machinery for constructing the "inductive limit" of p-energy forms on p.-c.f. self-similar structures can be stated in Theorems 6.21 and 6.22 below, which are extensions of the counterpart for resistance forms given in [Kaj, Lemma 2.24, Theorem 2.25 and Corollary 2.43]. This approach will be used in Subsection 8.3, where the construction of p-energy forms due to [CGQ22] is reviewed. See also [Kig01, Sections 2.2, 2.3 and 3.3] for the details in the case $p = 2$.

Definition 6.20 (Compatible sequence of p-resistance forms on finite sets)**.** Let $\mathcal{V}_n$ be a non-empty finite set and let $\mathcal{E}^{(n)}$ be a p-resistance form on $\mathcal{V}_n$ for each $n \in \mathbb{N} \cup \{0\}$. We say that the sequence $\mathcal{S} := \{(\mathcal{V}_n, \mathcal{E}^{(n)})\}_{n \in \mathbb{N} \cup \{0\}}$ is a *compatible sequence of p-resistance forms* if and only if $\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$ and $\mathcal{E}^{(n+1)}|_{\mathcal{V}_n} = \mathcal{E}^{(n)}$ for any $n \in \mathbb{N} \cup \{0\}$.

Theorem 6.21. *Let $\mathcal{S} = \{(\mathcal{V}_n, \mathcal{E}^{(n)})\}_{n \in \mathbb{N} \cup \{0\}}$ be a compatible sequence of p-resistance forms. We define $\mathcal{V}_* := \bigcup_{n \in \mathbb{N} \cup \{0\}} \mathcal{V}_n$,*

$$\mathcal{F}_{\mathcal{S}} := \Bigl\{ u \in \mathbb{R}^{\mathcal{V}_*} \Bigm| \lim_{n \to \infty} \mathcal{E}^{(n)}(u|_{\mathcal{V}_n}) < \infty \Bigr\}, \quad \textit{and} \tag{6.19}$$

$$\mathcal{E}_{\mathcal{S}}(u) := \lim_{n \to \infty} \mathcal{E}^{(n)}(u|_{\mathcal{V}_n}), \quad u \in \mathcal{F}_{\mathcal{S}}. \tag{6.20}$$

Then $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$ is a p-resistance form on $\mathcal{V}_*$ and $\mathcal{E}_{\mathcal{S}}|_{\mathcal{V}_n} = \mathcal{E}^{(n)}$ for any $n \in \mathbb{N} \cup \{0\}$.

Proof. Noting that $\{\mathcal{E}^{(n)}(u|_{\mathcal{V}_n})\}_{n \in \mathbb{N} \cup \{0\}}$ is non-decreasing for any $u \in \mathbb{R}^{\mathcal{V}_*}$, we easily obtain (RF1)$_p$ for $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$. To see (RF5)$_p$ for $(\mathcal{E}_{\mathcal{S},*}, \mathcal{F}_{\mathcal{S}})$, fix $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0, p]$, $q_2 \in [p, \infty]$ and $T = (T_1, \dots, T_{n_2}) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfying (2.1). Let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}_{\mathcal{S}}^{n_1}$. Then, for any $l \in \{1, \dots, n_2\}$, (GC)$_p$ for $\mathcal{E}^{(n)}$ implies that

$$\begin{aligned}\mathcal{E}^{(n)}(T_l(\boldsymbol{u})|_{\mathcal{V}_n})^{1/p} &\leq \bigl\| \bigl(\mathcal{E}^{(n)}(T_l(\boldsymbol{u}|_{\mathcal{V}_n}))^{1/p}\bigr)_{l=1}^{n_2} \bigr\|_{\ell^{q_2}} \\ &\leq \bigl\| \bigl(\mathcal{E}^{(n)}(u_k|_{\mathcal{V}_n})^{1/p}\bigr)_{k=1}^{n_1} \bigr\|_{\ell^{q_1}} \leq \bigl\| \bigl(\mathcal{E}_{\mathcal{S},*}(u_k)^{1/p}\bigr)_{k=1}^{n_1} \bigr\|_{\ell^{q_1}} < \infty.\end{aligned}$$

By letting $n \to \infty$, we obtain (GC)$_p$ for $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$, i.e., (RF5)$_p$ for $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$ holds. Before proving (RF2)$_p$-(RF4)$_p$ for $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$, we shall show the following claim:

$$\begin{array}{l}\text{For any } n \in \mathbb{N} \cup \{0\} \text{ and any } u \in \mathbb{R}^{\mathcal{V}_n}, \text{ there exists a unique } h^{\mathcal{S}}_{\mathcal{V}_n}[u] \in \mathcal{F}_{\mathcal{S}} \text{ such} \\ \text{that } h^{\mathcal{S}}_{\mathcal{V}_n}[u]\big|_{\mathcal{V}_n} = u \text{ and } \mathcal{E}_{\mathcal{S}}\bigl(h^{\mathcal{S}}_{\mathcal{V}_n}[u]\bigr) = \min\{\mathcal{E}_{\mathcal{S}}(v) \mid v \in \mathcal{F}_{\mathcal{S}}, v|_{\mathcal{V}_n} = u\} = \mathcal{E}^{(n)}(u).\end{array} \tag{6.21}$$

To prove (6.21), by (RF1)$_p$ and (RF5)$_p$ for $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$, we first note that $\#\{v \in \mathcal{F}_{\mathcal{S}} \mid \mathcal{E}_{\mathcal{S}}(v) = \alpha\} \leq 1$, where $\alpha := \min\{\mathcal{E}_{\mathcal{S}}(v) \mid v \in \mathcal{F}_{\mathcal{S}}, v|_{\mathcal{V}_n} = u\}$. (Recall the arguments in (6.10) and (6.11).) Hence it suffices to show the existence of the minimizer realizing α. For any $k_2 \geq k_1 \geq n$, we have $h^{\mathcal{E}^{(k_2)}}_{\mathcal{V}_n}[u]\big|_{\mathcal{V}_{k_1}} = h^{\mathcal{E}^{(k_1)}}_{\mathcal{V}_n}[u]$ by $\mathcal{E}^{(k_2)}|_{\mathcal{V}_{k_1}} = \mathcal{E}^{(k_1)}$ and Proposition 6.15, which implies that $u_*(x) := h^{\mathcal{E}^{(k)}}_{\mathcal{V}_n}[u](x)$ for $x \in \mathcal{V}_k$ with $k \geq n$ is well-defined. Clearly, $u_*|_{\mathcal{V}_n} = u$. For any $k \geq n$, we have $\mathcal{E}^{(k)}(u_*|_{\mathcal{V}_k}) = \mathcal{E}^{(k+1)}(u_*|_{\mathcal{V}_{k+1}})$ by Proposition 6.15 again, whence $u_* \in \mathcal{F}_{\mathcal{S}}$ and $\mathcal{E}_{\mathcal{S}}(u_*) = \mathcal{E}^{(n)}(u)$. Since $\mathcal{E}^{(n)}(u) \leq \mathcal{E}_{\mathcal{S}}(v)$ for any $v \in \mathcal{F}_{\mathcal{S}}$ with $v|_{\mathcal{V}_n} = u$, we also get $\mathcal{E}_{\mathcal{S}}(u_*) = \alpha$, so $h^{\mathcal{S}}_{\mathcal{V}_n}[u] := u_*$ is the desired function.

Now let us go back to the proof of (RF2)$_p$-(RF4)$_p$.

(RF3)$_p$: This is immediate since $\mathcal{F}_{\mathcal{S}}|_{\mathcal{V}_n} = \mathbb{R}^{V_n}$ for any $n \in \mathbb{N} \cup \{0\}$ by (6.21).

(RF4)$_p$: Let $x, y \in \mathcal{V}_*$ with $x \neq y$ and let $n \in \mathbb{N} \cup \{0\}$ satisfy $x, y \in \mathcal{V}_n$. Let $u := h^{\mathcal{E}^n}_{\{x,y\}}\bigl[\mathbb{1}^{\{x,y\}}_x\bigr] \in \mathbb{R}^{\mathcal{V}_n}$. Then for any $v \in \mathcal{F}_{\mathcal{S}}$ with $v|_{\{x,y\}} = \mathbb{1}^{\{x,y\}}_x$,

$$\mathcal{E}_{\mathcal{S}}(v) \overset{(6.21)}{\geq} \mathcal{E}^{(n)}(v|_{\mathcal{V}_n}) \geq R_{\mathcal{E}^n}(x, y)^{-1} = \mathcal{E}^{(n)}(u) \overset{(6.21)}{=} \mathcal{E}_{\mathcal{S}}\bigl(h^{\mathcal{S}}_{\mathcal{V}_n}[u]\bigr).$$

Therefore, we have

$$R_{\mathcal{E}_{\mathcal{S}}}(x, y) = \mathcal{E}_{\mathcal{S}}\bigl(h^{\mathcal{S}}_{\mathcal{V}_n}[u]\bigr)^{-1} = R_{\mathcal{E}^{(n)}}(x, y) < \infty. \tag{6.22}$$

(RF2)$_p$: Fix $x_* \in \mathcal{V}_*$ and let $\{u_k\}_{k\in\mathbb{N}}$ be such that $u_k \in \mathcal{F}_{\mathcal{S}}$, $u_k(x_*) = 0$ and $\lim_{k\wedge l\to\infty} \mathcal{E}_{\mathcal{S}}(u_k - u_l) = 0$. From (RF4)$_p$, $\{u_k(x)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for any $x \in \mathcal{V}_*$, so we define $u \in \mathbb{R}^{V_*}$ by $u(x) := \lim_{k\to\infty} u_k(x)$. For any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $\sup_{k,l\geq N_0} \mathcal{E}_{\mathcal{S}}(u_k - u_l) \leq \varepsilon$. Since $\mathcal{E}^{(n)}(\,\cdot\,)^{1/p}$ is a norm on the finite-dimensional vector space $\mathbb{R}^{\mathcal{V}_n}/\mathbb{R}\mathbb{1}_{\mathcal{V}_n}$, we obtain

$$\mathcal{E}^{(n)}(u|_{\mathcal{V}_n} - u_l|_{\mathcal{V}_n}) \leq \liminf_{k\to\infty} \mathcal{E}_{\mathcal{S}}(u_k - u_l) \leq \varepsilon \quad \text{for any } l \geq N_0 \text{ and any } n \in \mathbb{N}\cup\{0\}.$$

Since $n \in \mathbb{N}\cup\{0\}$ is arbitrary, we conclude that $u \in \mathcal{F}_{\mathcal{S}}$ and $\lim_{l\to\infty} \mathcal{E}_{\mathcal{S}}(u - u_l) = 0$, which proves that $(\mathcal{F}_{\mathcal{S}}/\mathbb{R}\mathbb{1}_{\mathcal{V}_*}, \mathcal{E}_{\mathcal{S}}^{1/p})$ is a Banach space.

Now we know that $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$ is a p-resistance form on $\mathcal{V}_*$. Then (6.21) means that $h^{\mathcal{S}}_{\mathcal{V}_n} = h^{\mathcal{E}_{\mathcal{S}}}_{\mathcal{V}_n}[u]$ for any $u \in \mathbb{R}^{\mathcal{V}_n}$, whence $\mathcal{E}_{\mathcal{S}}|_{\mathcal{V}_n} = \mathcal{E}^{(n)}$ by (6.21) again. □

The following theorem yields a p-resistance form on the completion of $(X, R_{\mathcal{E}}^{1/p})$.

Theorem 6.22. *Let $(\widehat{X}, \widehat{d})$ be the completion of the metric space $(X, R_{\mathcal{E}}^{1/p})$. We define $\widehat{\mathcal{F}} \subseteq \mathbb{R}^{\widehat{X}}$ and $\widehat{\mathcal{E}}\colon \widehat{\mathcal{F}} \to [0,\infty)$ by*

$$\widehat{\mathcal{F}} := \bigl\{u \in C(\widehat{X}) \bigm| u|_X \in \mathcal{F}\bigr\}, \tag{6.23}$$

$$\widehat{\mathcal{E}}(u) := \mathcal{E}(u|_X), \quad u \in \widehat{\mathcal{F}}. \tag{6.24}$$

Then $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ is a p-resistance form on $\widehat{X}$, $R_{\widehat{\mathcal{E}}}^{1/p} = \widehat{d}$, and the map $\widehat{\mathcal{F}} \ni u \mapsto u|_X \in \mathcal{F}$ is a linear isomorphism.

Proof. Set $\widehat{R}(x,y) := \widehat{d}(x,y)^p$ for convenience, then $\widehat{R}\,\big|_{X\times X} = R_{\mathcal{E}}$. For any $u \in \mathcal{F}$, we know that u is uniformly continuous with respect to $\widehat{d}$ by (6.3) for $(\mathcal{E},\mathcal{F})$, so there exists a unique $\widehat{u} \in C(\widehat{X})$ satisfying $\widehat{u}|_X = u$ and then $\widehat{u} \in \widehat{\mathcal{F}}$. This implies that the map $\widehat{\mathcal{F}} \ni u \mapsto u|_X \in \mathcal{F}$ is a bijection and thus it is a linear isomorphism. Also, for $u \in \widehat{\mathcal{F}}$, we define the continuous function $\eta_u\colon \widehat{X}\times\widehat{X} \to \mathbb{R}$ by $\eta_u(x,y) := |u(x) - u(y)|^p - \widehat{R}(x,y)\widehat{\mathcal{E}}(u)$. Since $\eta_u|_{X\times X} \leq 0$ by (6.3) for $R_{\mathcal{E}}$, the continuity of η_u yields

$$|u(x) - u(y)|^p \leq \widehat{R}(x,y)\widehat{\mathcal{E}}(u), \quad x, y \in \widehat{X}\times\widehat{X}. \tag{6.25}$$

Now we show (RF1)$_p$-(RF5)$_p$ for $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$.

(RF1)$_p$: Clearly, $\widehat{\mathcal{F}}$ is a linear subspace of $\mathbb{R}^{\widehat{X}}$ containing $\mathbb{R}\mathbb{1}_{\widehat{X}}$ and $\widehat{\mathcal{E}}(\,\cdot\,)^{1/p}$ is a seminorm on $\widehat{\mathcal{F}}$. By $\mathbb{1}_{\widehat{X}}|_X = \mathbb{1}_X$ and (RF1)$_p$ for $(\mathcal{E},\mathcal{F})$, it holds that $\{u \in \widehat{\mathcal{F}} \mid \widehat{\mathcal{E}}(u) = 0\} = \mathbb{R}\mathbb{1}_{\widehat{X}}$.

(RF2)$_p$: This is immediate from (RF2)$_p$ for $(\mathcal{E},\mathcal{F})$ since $\widehat{\mathcal{F}} \ni u \mapsto u|_X \in \mathcal{F}$ is a linear isomorphism.

(RF5)$_p$: This is immediate from (RF5)$_p$ for $(\mathcal{E},\mathcal{F})$.

(RF3)$_p$ and (RF4)$_p$: Let $x, y \in \widehat{X}$ with $x \neq y$ and let $\{x_n\}_{n\geq 0}, \{y_n\}_{n\geq 0} \subseteq X$ satisfy $\lim_{n\to\infty} \widehat{R}(x, x_n) = \lim_{n\to\infty} \widehat{R}(y, y_n) = 0$. We can assume that $x_n \neq y_n$ for any $n \geq 0$.

Let $u_n \in \widehat{\mathcal{F}}$ be the unique function satisfying $u_n|_X = h^{\mathcal{E}}_{\{x_n,y_n\}}[\mathbb{1}^{\{x_n,y_n\}}_{x_n}]$. Then $\{\widehat{\mathcal{E}}(u_n)\}_{n\geq 0}$ is bounded in $[0,\infty)$ since $\widehat{\mathcal{E}}(u_n) = R_{\mathcal{E}}(x_n,y_n)^{-1} = \widehat{R}(x_n,y_n)^{-1} \to \widehat{R}(x,y)^{-1}$ as $n \to \infty$. Also, it is easy to see that $0 \leq u_n \leq 1$. From (6.25) and the Arzelá–Ascoli theorem, there exist a subsequence $\{u_{n_k}\}_k$ and $u_* \in C(\widehat{X})$ such that $\lim_{k\to\infty} \|u_* - u_{n_k}\|_{\sup,\widehat{X}} = 0$. A similar argument as in the proof of (RF2)$_p$ for $(\mathcal{E}_{\mathcal{S}},\mathcal{F}_{\mathcal{S}})$ in Theorem 6.21 implies that $u_* \in \widehat{\mathcal{F}}$ and $\lim_{k\to\infty} \widehat{\mathcal{E}}(u_* - u_{n_k}) = 0$. Now we define $u \in \widehat{\mathcal{F}}$ by $u := u_* - u_*(y)$ so that $u(y) = 0$. Then we have from (6.25) that

$$|u(x_{n_k}) - u(y_{n_k}) - 1|^p \leq \widehat{R}(x_{n_k},y_{n_k})\widehat{\mathcal{E}}(u - u_{n_k}) \xrightarrow[k\to\infty]{} 0,$$

whence $u(x) = 1$, in particular, (RF3)$_p$ holds. By (6.25) again, we obtain $R_{\widehat{\mathcal{E}}}(x,y) \leq \widehat{R}(x,y) < \infty$, so (RF4)$_p$ holds. Moreover, this also shows $R_{\widehat{\mathcal{E}}}(x,y) = \widehat{R}(x,y) = \widehat{\mathcal{E}}(u)^{-1}$. □

Corollary 6.23. *Let $\mathcal{S} = \{(\mathcal{V}_n, \mathcal{E}^{(n)})\}_{n\in\mathbb{N}\cup\{0\}}$ be a compatible sequence of p-resistance forms and let (K,d) be the completion of $(\mathcal{V}_*, R^{1/p}_{\mathcal{E}_{\mathcal{S}}})$, where $(\mathcal{E}_{\mathcal{S}},\mathcal{F}_{\mathcal{S}})$ is the p-resistance form on $\mathcal{V}_* = \bigcup_{n\in\mathbb{N}\cup\{0\}} \mathcal{V}_n$ given in Theorem 6.21. We define $\widehat{\mathcal{F}}_{\mathcal{S}} \subseteq \mathbb{R}^K$ and $\widehat{\mathcal{E}}_{\mathcal{S}} \colon \widehat{\mathcal{F}}_{\mathcal{S}} \to [0,\infty)$ by*

$$\widehat{\mathcal{F}}_{\mathcal{S}} := \bigl\{u \in C(K) \bigm| u|_{\mathcal{V}_*} \in \mathcal{F}_{\mathcal{S}}\bigr\} = \Bigl\{u \in C(K) \Bigm| \lim_{n\to\infty} \mathcal{E}^{(n)}(u|_{\mathcal{V}_n}) < \infty\Bigr\}, \tag{6.26}$$

$$\widehat{\mathcal{E}}_{\mathcal{S}}(u) := \mathcal{E}_{\mathcal{S}}(u|_{\mathcal{V}_*}) = \lim_{n\to\infty} \mathcal{E}^{(n)}(u|_{\mathcal{V}_n}), \quad u \in \widehat{\mathcal{F}}_{\mathcal{S}}. \tag{6.27}$$

Then $(\widehat{\mathcal{E}}_{\mathcal{S}},\widehat{\mathcal{F}}_{\mathcal{S}})$ is a p-resistance form on K, $R^{1/p}_{\widehat{\mathcal{E}}_{\mathcal{S}}} = d$, and the map $\widehat{\mathcal{F}}_{\mathcal{S}} \ni u \mapsto u|_{\mathcal{V}_*} \in \mathcal{F}_{\mathcal{S}}$ is a linear isomorphism. In particular, $\widehat{\mathcal{E}}_{\mathcal{S}}\big|_{\mathcal{V}_n} = \mathcal{E}^{(n)}$ for any $n \in \mathbb{N}\cup\{0\}$.

Proof. We obtain the desired assertions by applying Theorem 6.22 with $\mathcal{V}_*$, $(\mathcal{E}_{\mathcal{S}},\mathcal{F}_{\mathcal{S}})$ in place of X, $(\mathcal{E},\mathcal{F})$. Also, by $\mathcal{E}_{\mathcal{S}}|_{\mathcal{V}_n} = \mathcal{E}^{(n)}$ (see Theorem 6.21) and the fact that $\widehat{\mathcal{F}}_{\mathcal{S}} \ni u \mapsto u|_{\mathcal{V}_*} \in \mathcal{F}_{\mathcal{S}}$ is a linear isomorphism, we have $\widehat{\mathcal{E}}_{\mathcal{S}}\big|_{\mathcal{V}_n} = \mathcal{E}^{(n)}$. □

We conclude this subsection with the strong locality of p-resistance forms.

Definition 6.24 (Strong locality for p-resistance form)**.** We say that $(\mathcal{E},\mathcal{F})$ is *strongly local* if and only if $\mathcal{E}(u_1;v) = \mathcal{E}(u_2;v)$ for any $u_1,u_2,v \in \mathcal{F}$ with either $\mathrm{supp}_X[u_1 - u_2 - a\mathbb{1}_X]$ or $\mathrm{supp}_X[v - b\mathbb{1}_X]$ is compact and $(u_1(x) - u_2(x) - a)(v(x) - b) = 0$ for any $x \in X$ for some $a,b \in \mathbb{R}$.

In the following proposition, we discuss relations among the strong locality in Definition 6.24, (SL1) and (SL2) of $(\mathcal{E},\mathcal{F})$.

Proposition 6.25. (a) *If $(\mathcal{E},\mathcal{F})$ is strongly local (in the sense of Definition 6.24), then $(\mathcal{E},\mathcal{F})$ satisfies the strongly local property (SL2).*

(b) *If $(\mathcal{E},\mathcal{F})$ is regular and strongly local (in the sense of Definition 6.24), then for any $u_1, u_2, v \in \mathcal{F}$ with either* $\operatorname{supp}_X[u_1 - a_1\mathbb{1}_X]$ *or* $\operatorname{supp}_X[u_2 - a_2\mathbb{1}_X]$ *compact and $(u_1(x) - a_1)(u_2(x) - a_2) = 0$ for any $x \in X$ for some $a_1, a_2 \in \mathbb{R}$,*

$$\mathcal{E}(u_1 + u_2 + v) + \mathcal{E}(v) = \mathcal{E}(u_1 + v) + \mathcal{E}(u_2 + v).$$

In particular, $(\mathcal{E},\mathcal{F})$ satisfies the strongly local property (SL1).

(c) *Assume that $(\mathcal{E},\mathcal{F})$ is regular and satisfies the strongly local property (SL2). Then $(\mathcal{E},\mathcal{F})$ is strongly local (in the sense of Definition 6.24).*

Proof. (a): If $u_1, u_2, v \in \mathcal{F}$ and $a, b \in \mathbb{R}$ satisfy $\operatorname{supp}_X[u_1 - u_2 - a\mathbb{1}_X] \cap \operatorname{supp}_X[v - b\mathbb{1}_X] = \emptyset$, then it is immediate that $(u_1(x) - u_2(x) - a)(v(x) - b) = 0$ for any $x \in X$. Hence $(\mathcal{E},\mathcal{F})$ satisfies the strongly local property (SL2).

(b): Let $\varphi_n \in C(\mathbb{R})$ be given by $\varphi_n(t) \coloneqq t - (-\frac{1}{n}) \vee (t \wedge \frac{1}{n})$ for each $n \in \mathbb{N}$. Set $u_{1,n}(x) \coloneqq \varphi_n(u_1(x) - a_1)$ and $u_{2,n}(x) \coloneqq \varphi_n(u_2(x) - a_2)$. Then $u_{i,n} \in \mathcal{F}$ and $\lim_{n\to\infty} \mathcal{E}(u_i - u_{i,n}) = 0$ for $i \in \{1, 2\}$ by Corollary 6.18-(a) and (RF1)_p. Furthermore, $\operatorname{supp}_X[u_{1,n}] \cap \operatorname{supp}_X[u_{2,n}] = \emptyset$ and either $\operatorname{supp}_X[u_{1,n}]$ or $\operatorname{supp}_X[u_{2,n}]$ is compact. By (a), Proposition 3.30-(b) and Proposition 6.6, we have $\mathcal{E}(u_{1,n} + u_{2,n} + v) + \mathcal{E}(v) = \mathcal{E}(u_{1,n} + v) + \mathcal{E}(u_{2,n} + v)$ for any $v \in \mathcal{F}$. We obtain the desired assertion by letting $n \to \infty$.

(c): Set $v_n(x) \coloneqq \varphi_n(v(x) - b)$ for each $n \in \mathbb{N}$, where φ_n is the same as in the proof of (b). Then $v_n \in \mathcal{F}$ and $\lim_{n\to\infty} \mathcal{E}(v - v_n) = 0$ by Corollary 6.18-(a) and (RF1)_p. Furthermore, $\operatorname{supp}_X[u_1 - u_2 - a\mathbb{1}_X] \cap \operatorname{supp}_X[v_n] = \emptyset$ and either $\operatorname{supp}_X[u_1 - u_2 - a\mathbb{1}_X]$ or $\operatorname{supp}_X[v_n]$ is compact. Hence, by (SL2), it follows that $\mathcal{E}(u_1; v_n) = \mathcal{E}(u_2; v_n)$. We obtain $\mathcal{E}(u_1; v) = \mathcal{E}(u_2; v)$ by letting $n \to \infty$. □

6.3 Weak comparison principles

In this subsection, we show some weak comparison principles in this context. The first one is an application of the strong subadditivity.

Proposition 6.26 (Weak comparison principle I)**.** *Let B be a non-empty subset of X. Then, for any $u, v \in \mathcal{F}|_B$ satisfying $u(y) \le v(y)$ for any $y \in B$, it holds that*

$$h_B^{\mathcal{E}}[u](x) \le h_B^{\mathcal{E}}[v](x) \quad \text{for any } x \in X. \tag{6.28}$$

In particular,

$$\inf_B u \le h_B^{\mathcal{E}}[u](x) \le \sup_B u \quad \text{for any } x \in X. \tag{6.29}$$

Proof. Let $f \coloneqq h_B^{\mathcal{E}}[u]$ and $g \coloneqq h_B^{\mathcal{E}}[v]$. We will prove $f \wedge g = f$, which immediately implies (6.28). Since $(f \wedge g)|_B = u$ and $(f \vee g)|_B = v$, we have

$$\mathcal{E}(f) \le \mathcal{E}(f \wedge g) \quad \text{and} \quad \mathcal{E}(g) \le \mathcal{E}(f \vee g).$$

By the strong subadditivity in (2.5), we obtain $\mathcal{E}(f \wedge g) = \mathcal{E}(f)$ (and $\mathcal{E}(f \vee g) = \mathcal{E}(g)$), which together with the uniqueness in Theorem 6.13, we have $f \wedge g = f$. □

We can extend the weak comparison principle above to arbitrary open subsets (see Proposition 6.30 below) if $(\mathcal{E},\mathcal{F})$ is regular and strongly local. This version of weak comparison principle will be used to prove the *strong comparison principle* on p.-c.f. self-similar structures in a forthcoming paper [KS.b]. We begin with some preparations.

Definition 6.27. Let U be a non-empty open subset of X.

(1) Define

$$\mathcal{F}_{\mathrm{loc}}(U) \coloneqq \left\{ f \in \mathbb{R}^U \;\middle|\; \begin{array}{l} f\mathbb{1}_V = f^{\#}\mathbb{1}_V \text{ for some } f^{\#} \in \mathcal{F} \text{ for each} \\ \text{relatively compact open subset } V \text{ of } U \end{array} \right\}.$$

(2) Assume that $(\mathcal{E},\mathcal{F})$ is strongly local. Let $V \subseteq U$ be an open subset. A function $h \in \mathcal{F}_{\mathrm{loc}}(U)$ is said to be $\mathcal{E}$-harmonic on V if $\mathcal{E}(h^{\#};\varphi) = 0$ for any $\varphi \in \mathcal{F}^0(V)$ with $\mathrm{supp}[\varphi]$ compact (with respect to the metric topology of $R_{\mathcal{E}}^{1/p}$), where $h^{\#} \in \mathcal{F}$ satisfies $h\mathbb{1}_{\mathrm{supp}[\varphi]} = h^{\#}\mathbb{1}_{\mathrm{supp}[\varphi]}$.

Remark 6.28. (1) If $X =: K$ comes from a self-similar structure and the topology induced by $R_{\mathcal{E}}^{1/p}$ coincides with the original topology of K, then the definition of $\mathcal{F}_{\mathrm{loc}}(U)$ above is the same as (5.30) by virtue of $\mathcal{F} \subseteq C(K)$.

(2) By the strong locality of $(\mathcal{E},\mathcal{F})$, the value $\mathcal{E}(h^{\#};\varphi)$ is independent of a particular choice of $h^{\#}$.

We need the following proposition to achieve the desired weak comparison principle.

Proposition 6.29. *Assume that X is locally compact and that $(\mathcal{E},\mathcal{F})$ is regular and strongly local. Let U be a non-empty open subset of X and let $u \in \mathcal{F}$ satisfy $u(x) = 0$ for any $x \in \partial_X U = \overline{U}^X \setminus U$. Then $u\mathbb{1}_U \in \mathcal{F}$.*

Proof. Define $\varphi_n \in C(\mathbb{R})$ by $\varphi_n(t) \coloneqq t - \left(\frac{1}{n}\right) \vee \left(t \wedge \frac{1}{n}\right)$ and set $A_n \coloneqq U \cap \mathrm{supp}_X[\varphi_n(u)]$ for each $n \in \mathbb{N}$. Since $u|_{\partial U} = 0$, $A_n = \overline{U}^X \cap \mathrm{supp}_X[\varphi_n(u)]$ and thus A_n is a compact subset of U. By Proposition 6.6, there exists $v_n \in \mathcal{F}$ such that $\mathbb{1}_{A_n} \leq v_n \leq \mathbb{1}_U$. Then we easily obtain $\varphi_n(u)\mathbb{1}_U = \varphi_n(u)v_n$, hence by Corollary 6.18-(a) and Proposition 2.2-(d) we have $\varphi_n(u)\mathbb{1}_U \in \mathcal{F}$. By the strong locality and Corollary 6.18-(a), $\{\varphi_n(u)\mathbb{1}_U\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{F}/\mathbb{R}\mathbb{1}_X, \mathcal{E}^{1/p})$. Thus, by (RF2)$_p$ and (6.3), $\{\varphi_n(u)\mathbb{1}_U\}_{n\in\mathbb{N}}$ converges in norm in $(\mathcal{F}/\mathbb{R}\mathbb{1}_X, \mathcal{E}^{1/p})$ to its pointwise limit $u\mathbb{1}_U$, whence $u\mathbb{1}_U \in \mathcal{F}$. □

Now we can state the desired version of the weak comparison principle.

Proposition 6.30 (Weak comparison principle II)**.** *Assume that X is locally compact and that $(\mathcal{E},\mathcal{F})$ is regular and strongly local. Let U be non-empty open subset of X such that $\overline{U}^X$ is compact and $U \neq X$. If $u, v \in C(\overline{U}^X) \cap \mathcal{F}_{\mathrm{loc}}(U)$ are $\mathcal{E}$-harmonic on U and $u(x) \leq v(x)$ for any $x \in \partial_X U = \overline{U}^X \setminus U$, then $u(x) \leq v(x)$ for any $x \in \overline{U}^X$.*

Proof. We first observe that $\partial_X O \neq \emptyset$ for any non-empty open subset of X such that $\overline{O}^X$ is compact and $O \neq X$. To this end, suppose that $\partial_X O = \emptyset$ and then show $O = X$.

We see from Proposition 3.26 that there exists $\varphi \in \mathcal{F} \cap C_c(X)$ such that $\varphi|_O = 1$ and $\varphi|_{X \setminus O} = 0$ since $O = \overline{O}^X$ is compact. By the strong locality for $(\mathcal{E}, \mathcal{F})$ and (RF1)$_p$, we have $\mathcal{E}(\varphi) = 0$ and hence $\varphi \in \mathbb{R}\mathbb{1}_X$. Therefore, $X \setminus O = \emptyset$ since O is non-empty.

Let us go back to the proof. Since u and v are uniformly continuous on $\overline{U}^X$ and $\partial_X U \neq \emptyset$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$V := \left\{ x \in U \,\middle|\, \operatorname{dist}_{R_{\mathcal{E}}^{1/p}}(x, \partial_X U) > \delta \right\} \neq \emptyset,$$

and $u(x) \leq v(x) + \varepsilon$ for any $x \in \overline{U}^X \setminus V$. Then V is a relatively compact open subset of U and hence there exist $u^{\#}, v^{\#} \in \mathcal{F}$ such that $u\mathbb{1}_V = u^{\#}\mathbb{1}_V$ and $v\mathbb{1}_V = v^{\#}\mathbb{1}_V$. Define $f := u^{\#} - (u^{\#} - v^{\#})^{+}\mathbb{1}_{X \setminus V}$, $g := v^{\#} + (u^{\#} - v^{\#})^{+}\mathbb{1}_{X \setminus V}$. Then $f, g \in \mathcal{F}$ by $u^{\#}(x) \leq v^{\#}(x)$ for any $x \in \partial_X V \neq \emptyset$, Propositions 2.2-(b) and 6.29. We also have $f, g \in \mathcal{H}_{\mathcal{E}, X \setminus V}$ by the strong locality for $(\mathcal{E}, \mathcal{F})$. Since $f(x) = (u^{\#} \wedge v^{\#})(x) \leq (u^{\#} \vee v^{\#})(x) = g(x)$ for any $x \in X \setminus V$, Proposition 6.26 implies that $u(x) = u^{\#}(x) = f(x) \leq g(x) = v^{\#}(x) = v(x)$ for any $x \in V$. Therefore, we conclude that $u(x) \leq v(x) + \varepsilon$ for any $x \in \overline{U}^X$. Since $\varepsilon > 0$ is arbitrary, we complete the proof. □

6.4 Hölder regularity of harmonic functions

In this subsection, we present a sharp Hölder regularity estimate on $\mathcal{E}$-harmonic functions and prove that $R_{\mathcal{E}}^{1/(p-1)}$ is a metric on X.

As an application of Proposition 3.9, we can show the following Hölder continuity estimate for $\mathcal{E}$-harmonic functions.

Theorem 6.31. *Let B be a non-empty subset of X. Then for any $x \in X \setminus B^{\mathcal{F}}$ and any $y \in X \setminus \{x\}$,*

$$h^{\mathcal{E}}_{B \cup \{x\}}\bigl[\mathbb{1}_B^{B \cup \{x\}}\bigr](y) \leq \frac{R_{\mathcal{E}}(x, y)^{1/(p-1)}}{R_{\mathcal{E}}(x, B)^{1/(p-1)}}. \tag{6.30}$$

Moreover, for any $h \in \mathcal{H}_{\mathcal{E}, B}$ with $\sup_B |h| < \infty$, any $x \in X \setminus B^{\mathcal{F}}$ and any $y \in X$,

$$|h(x) - h(y)| \leq \frac{R_{\mathcal{E}}(x, y)^{1/(p-1)}}{R_{\mathcal{E}}(x, B)^{1/(p-1)}} \operatorname*{osc}_B[h]. \tag{6.31}$$

Proof. To show (6.30), on one hand, we see that

$$\begin{aligned} -\mathcal{E}|_{B \cup \{x\}}(\mathbb{1}_B; \mathbb{1}_x) &= \mathcal{E}|_{B \cup \{x\}}(\mathbb{1}_B; \mathbb{1}_{B \cup \{x\}}) - \mathcal{E}|_{B \cup \{x\}}(\mathbb{1}_B; \mathbb{1}_x) \\ &= \mathcal{E}|_{B \cup \{x\}}(\mathbb{1}_B; \mathbb{1}_B) = R_{\mathcal{E}}(x, B)^{-1}. \end{aligned} \tag{6.32}$$

On the other hand,

$$\begin{aligned} &-\mathcal{E}|_{B \cup \{x\}}(\mathbb{1}_B; \mathbb{1}_x) \\ &\quad = -\mathcal{E}\Bigl(h^{\mathcal{E}}_{B \cup \{x\}}[\mathbb{1}_B]; h^{\mathcal{E}}_{B \cup \{x,y\}}[\mathbb{1}_x]\Bigr) \end{aligned}$$

$$
\begin{aligned}
&= -\mathcal{E}|_{B\cup\{x,y\}}\Bigl(h^{\mathcal{E}}_{B\cup\{x\}}[\mathbb{1}_B]\big|_{B\cup\{x,y\}};\mathbb{1}_x\Bigr)\\
&\geq -\mathcal{E}|_{B\cup\{x,y\}}\Bigl(\bigl(h^{\mathcal{E}}_{B\cup\{x\}}[\mathbb{1}_B](y)\cdot h^{\mathcal{E}}_{\{x,y\}}[\mathbb{1}_y]\bigr)\big|_{B\cup\{x,y\}};\mathbb{1}_x\Bigr) \quad \text{(by Proposition 3.9)}\\
&= -h^{\mathcal{E}}_{B\cup\{x\}}[\mathbb{1}_B](y)^{p-1}\mathcal{E}|_{B\cup\{x,y\}}\Bigl(h^{\mathcal{E}}_{\{x,y\}}[\mathbb{1}_y]\big|_{B\cup\{x,y\}};\mathbb{1}_x\Bigr) = h^{\mathcal{E}}_{B\cup\{x\}}[\mathbb{1}_B](y)^{p-1}R_{\mathcal{E}}(x,y)^{-1}.
\end{aligned} \tag{6.33}
$$

Here, we used $\mathbb{1}_{\{x,y\}} - h^{\mathcal{E}}_{\{x,y\}}[\mathbb{1}_y] = h^{\mathcal{E}}_{\{x,y\}}[\mathbb{1}_x]$ (see Remark 6.14) in the last equality. We obtain (6.30) by combining (6.32) and (6.33).

Next we prove (6.31). Let $x \in X \setminus B^{\mathcal{F}}$, $y \in X$ and $h \in \mathcal{H}_{\mathcal{E},B}$ with $\sup_B |h| < \infty$. We can assume that $x \neq y$. Then we see that

$$
\begin{aligned}
h - h(x) &\leq h^{\mathcal{E}}_{B\cup\{x\}}\Bigl[(h-h(x))^+\big|_{B\cup\{x\}}\Bigr] \quad \text{(by Propositions 6.26 and 6.15)}\\
&\leq h^{\mathcal{E}}_{B\cup\{x\}}\Bigl[\operatorname*{osc}_B[h]\cdot\mathbb{1}_B^{B\cup\{x\}}\Bigr] \quad \text{(by Proposition 6.26 and } (h-h(x))^+(x)=0)\\
&= \operatorname*{osc}_B[h]\cdot h^{\mathcal{E}}_{B\cup\{x\}}\Bigl[\mathbb{1}_B^{B\cup\{x\}}\Bigr].
\end{aligned}
$$

Similarly, we have

$$
h - h(x) \geq -h^{\mathcal{E}}_{B\cup\{x\}}\Bigl[(h-h(x))^-\big|_{B\cup\{x\}}\Bigr] \geq -\operatorname*{osc}_B[h]\cdot h^{\mathcal{E}}_{B\cup\{x\}}\Bigl[\mathbb{1}_B^{B\cup\{x\}}\Bigr].
$$

Hence, by combining with (6.30), we get (6.31). □

Using Theorem 6.31, we can show the triangle inequality for $R_{\mathcal{E}}^{1/(p-1)}$.

Corollary 6.32. *$R_{\mathcal{E}}^{1/(p-1)}\colon X\times X \to [0,\infty)$ is a metric on X.*

Definition 6.33 (p-Resistance metric)**.** We define $\widehat{R}_{p,\mathcal{E}} := R_{\mathcal{E}}^{1/(p-1)}$. We call $\widehat{R}_{p,\mathcal{E}}$ the *p-resistance metric* of $(\mathcal{E},\mathcal{F})$.

Proof of Corollary 6.32. It suffices to prove $R_{\mathcal{E}}(x,z)^{1/(p-1)} \leq R_{\mathcal{E}}(x,y)^{1/(p-1)} + R_{\mathcal{E}}(y,z)^{1/(p-1)}$ for any $x,y,z \in X$ with $\#\{x,y,z\} = 3$. By (6.30) with $B = \{z\}$, we have $h^{\mathcal{E}}_{\{x,z\}}\bigl[\mathbb{1}_x^{\{x,z\}}\bigr](y) \leq \frac{R_{\mathcal{E}}(x,y)^{1/(p-1)}}{R_{\mathcal{E}}(x,z)^{1/(p-1)}}$. By exchanging the roles of x and z, we get $h^{\mathcal{E}}_{\{x,z\}}\bigl[\mathbb{1}_z^{\{x,z\}}\bigr](y) \leq \frac{R_{\mathcal{E}}(y,z)^{1/(p-1)}}{R_{\mathcal{E}}(x,z)^{1/(p-1)}}$. Since $\mathbb{1}_X = h^{\mathcal{E}}_{\{x,z\}}\bigl[\mathbb{1}_x^{\{x,z\}}\bigr] + h^{\mathcal{E}}_{\{x,z\}}\bigl[\mathbb{1}_z^{\{x,z\}}\bigr]$, we have

$$
1 \leq \frac{R_{\mathcal{E}}(x,y)^{1/(p-1)}}{R_{\mathcal{E}}(x,z)^{1/(p-1)}} + \frac{R_{\mathcal{E}}(y,z)^{1/(p-1)}}{R_{\mathcal{E}}(x,z)^{1/(p-1)}},
$$

which proves the desired triangle inequality for $R_{\mathcal{E}}^{1/(p-1)}$. □

Example 6.34. Let $p \in (1,\infty)$ and $(\mathcal{E},\mathcal{F})$ be a p-resistance form on the unit open interval $(0,1)$ given by

$$
\mathcal{F} := W^{1,p}(0,1) \quad \text{and} \quad \mathcal{E}(u) := \int_0^1 |\nabla u|^p\,dx.
$$

(Recall Example 6.3-(1).) For any $x, y \in (0,1)$ with $0 < x < y < 1$, we easily see that $u \in W^{1,p}(0,1)$ defined by $u(t) \coloneqq (y-x)^{-1}(t-x)\mathbb{1}_{[x,y]}(t)$, $t \in (0,1)$, is $\mathcal{E}$-harmonic on $(0,1) \setminus \{x,y\}$. Therefore we have $R_{\mathcal{E}}(x,y) = (y-x)^{p-1}$ and the p-resistance metric $\widehat{R}_{p,\mathcal{E}}$ coincides with the Euclidean metric on $(0,1)$. In particular, the Hölder regularity estimate (6.31) is sharp. This example also shows that exponent $1/(p-1)$ in the p-resistance metric is sharp, that is, $R_{\mathcal{E}}^{\alpha}$ is not a metric for $\alpha > 1/(p-1)$ in general.

6.5 Elliptic Harnack inequality for non-negative harmonic functions

Throughout this subsection, we assume the existence of p-energy measures $\{\Gamma\langle u\rangle\}_{u\in\mathcal{F}}$ (dominated by $(\mathcal{E},\mathcal{F})$) satisfying $(\mathrm{Cla})_p$. For simplicity, set $\widehat{R}_p \coloneqq \widehat{R}_{p,\mathcal{E}} = R_{\mathcal{E}}^{1/(p-1)}$.

In this subsection, we establish the elliptic Harnack inequality for non-negative $\mathcal{E}$-superharmonic functions (Theorem 6.37) under some extra analytic conditions. The following two lemmas are key ingredients of the proof of Theorem 6.37.

Lemma 6.35 (Two-point estimate)**.** *Assume that there exist a Borel measure μ on X, $\beta, Q \in (0,\infty)$ with $\beta > Q$ and $A, C \in [1,\infty)$ such that for any $(x,s) \in X \times (0,\infty)$ and any $u \in \mathcal{F}$,*

$$0 < \mu\bigl(B_{\widehat{R}_p}(x,r)\bigr) \le C\left(\frac{r}{s}\right)^{Q} \mu\bigl(B_{\widehat{R}_p}(x,s)\bigr) \quad \text{for any } r \in [s,\infty), \tag{6.34}$$

and

$$\int_{B_{\widehat{R}_p}(x,s)} \left| u(y) - \fint_{B_{\widehat{R}_p}(x,s)} u\,d\mu \right|^p \mu(dy) \le C s^{\beta} \int_{B_{\widehat{R}_p}(x,As)} d\Gamma\langle u\rangle. \tag{6.35}$$

Then there exists $\widetilde{A}, \widetilde{C} \in [1,\infty)$ such that for any $(x,s) \in X \times (0,\infty)$, any $y, z \in B_{\widehat{R}_p}(x,s)$ and any $u \in \mathcal{F}$,

$$|u(y) - u(z)|^p \le \widetilde{C} \frac{s^{\beta}}{\mu\bigl(B_{\widehat{R}_p}(x,\widetilde{A}s)\bigr)} \int_{B_{\widehat{R}_p}(x,\widetilde{A}s)} d\Gamma\langle u\rangle. \tag{6.36}$$

Proof. The proof will be done by a standard telescopic argument (see, e.g., [HK98, Proof of Lemma 5.17]). For $y \in B_{\widehat{R}_p}(x,s)$ and $n \in \mathbb{N} \cup \{0\}$, set $B_{y,n} \coloneqq B_{\widehat{R}_p}(y, 2^{-n}s)$, $AB_{y,n} \coloneqq B_{\widehat{R}_p}(y, 2^{-n}As)$ and $u_{y,i} \coloneqq \fint_{B_{y,i}} u\,d\mu$. Then for any $n \in \mathbb{N}$,

$$\begin{aligned} |u_{y,n} - u_{y,0}| \le \sum_{i=1}^{n} |u_{y,i} - u_{y,i-1}| &\le \sum_{i=1}^{n} \left(\fint_{B_{y,i-1}} |u - u_{y,i}|^p \, d\mu \right)^{1/p} \\ &\overset{(6.34)}{\lesssim} \sum_{i=1}^{n} \left(\fint_{B_{y,i}} |u - u_{y,i}|^p \, d\mu \right)^{1/p} \\ &\overset{(6.35)}{\lesssim} \sum_{i=1}^{n} \left(\frac{(2^{-i}s)^{\beta}}{\mu(B_{y,i})} \int_{AB_{y,i}} d\Gamma\langle u\rangle \right)^{1/p}, \end{aligned}$$

where we used Hölder's inequality in the second inequality. Letting $n \to \infty$, we obtain

$$
\begin{aligned}
|u(y) - u_{y,0}| &\lesssim \sum_{i=1}^{\infty} \left(\frac{(2^{-i}s)^{\beta}}{\mu(B_{y,i})} \int_{AB_{y,i}} d\Gamma\langle u\rangle \right)^{1/p} \\
&\overset{(6.34)}{\lesssim} \left(\frac{s^{\beta}}{\mu(AB_{y,0})} \int_{AB_{y,0}} d\Gamma\langle u\rangle \right)^{1/p} \sum_{i=1}^{\infty} 2^{-i(\beta-Q)/p} \\
&\overset{(6.34)}{\lesssim} \left(\frac{s^{\beta}}{\mu(B_{\widehat{R}_p}(x,(A+1)s))} \int_{B_{\widehat{R}_p}(x,(A+1)s)} d\Gamma\langle u\rangle \right)^{1/p}. \qquad (6.37)
\end{aligned}
$$

Similarly, for any $y, z \in B_{\widehat{R}_p}(x,s)$, we have from (6.35) that

$$
\begin{aligned}
&|u_{y,0} - u_{z,0}| \\
&\leq \left| u_{y,0} - \fint_{B_{\widehat{R}_p}(x,2s)} u\,d\mu \right| + \left| u_{z,0} - \fint_{B_{\widehat{R}_p}(x,2s)} u\,d\mu \right| \\
&\leq \left(\fint_{B_{\widehat{R}_p}(y,s)} \left| u - \fint_{B_{\widehat{R}_p}(x,2s)} u\,d\mu \right|^p d\mu \right)^{1/p} + \left(\fint_{B_{\widehat{R}_p}(z,s)} \left| u - \fint_{B_{\widehat{R}_p}(x,2s)} u\,d\mu \right|^p d\mu \right)^{1/p} \\
&\overset{(6.34)}{\lesssim} \left(\fint_{B_{\widehat{R}_p}(x,2s)} \left| u - \fint_{B_{\widehat{R}_p}(x,2s)} u\,d\mu \right|^p d\mu \right)^{1/p} \\
&\overset{(6.34),(6.35)}{\lesssim} \left(\frac{s^{\beta}}{\mu\bigl(B_{\widehat{R}_p}(x,2As)\bigr)} \int_{B_{\widehat{R}_p}(x,2As)} d\Gamma\langle u\rangle \right)^{1/p}. \qquad (6.38)
\end{aligned}
$$

By (6.37) and (6.38),

$$
\begin{aligned}
|u(y) - u(z)|^p &\lesssim |u(y) - u_{y,0}|^p + |u_{y,0} - u_{z,0}|^p + |u(z) - u_{z,0}|^p \\
&\lesssim \frac{s^{\beta}}{\mu\bigl(B_{\widehat{R}_p}(x,2As)\bigr)} \int_{B_{\widehat{R}_p}(x,2As)} d\Gamma\langle u\rangle,
\end{aligned}
$$

which shows (6.36). □

Lemma 6.36 (Log-Caccioppoli type inequality)**.** *Assume that $\{\Gamma\langle u\rangle\}_{u\in\mathcal{F}}$ satisfies the chain rule (CL2). Then there exists $C \in (0,\infty)$ (depending only on p) such that for any $(x,s) \in X \times (0,\infty)$, any $\varepsilon > 0$ and any $u \in \mathcal{F}$ that is $\mathcal{E}$-superharmonic in $B_{\widehat{R}_p}(x,2s)$ with $u \geq 0$ and $\mathcal{E}(u) = \Gamma\langle u\rangle(X)$.*

$$
\int_{B_{\widehat{R}_p}(x,s)} d\Gamma\langle \Phi_{\varepsilon}(u)\rangle \leq C \inf\bigl\{\mathcal{E}(\varphi) \bigm| \varphi \in \mathcal{F}, \varphi|_{B_{\widehat{R}_p}(x,s)} = 1, \operatorname{supp}_X[\varphi] \subseteq B_{\widehat{R}_p}(x,2s)\bigr\}, \qquad (6.39)
$$

where $\Phi_{\varepsilon} \in C^1(\mathbb{R})$ is any function satisfying $\Phi_{\varepsilon}(x) = \log(x+\varepsilon) - \log\varepsilon$ for $x \in [0,\infty)$.

Proof. Let $\varphi \in \mathcal{F}$ satisfy $\varphi|_{B_{\widehat{R}_p}(x,s)} = 1$, $\mathrm{supp}_X[\varphi] \subseteq B_{\widehat{R}_p}(x,2s)$ and

$$\mathcal{E}(\varphi) = \inf\bigl\{\mathcal{E}(\varphi) \bigm| \varphi \in \mathcal{F}, \varphi|_{B_{\widehat{R}_p}(x,s)} = 1, \mathrm{supp}_X[\varphi] \subseteq B_{\widehat{R}_p}(x,2s)\bigr\},$$

which exists by Theorem 6.13. Let $\varepsilon > 0$ and set $u_\varepsilon \coloneqq u + \varepsilon$. Note that $\varphi^p u_\varepsilon^{1-p} \in \mathcal{F}$ by Proposition 2.2-(d) and Corollary 2.4-(a). We see that

$$\begin{aligned}
&\int_{B_{\widehat{R}_p}(x,s)} d\Gamma\langle\Phi_\varepsilon(u)\rangle \leq \int_{B_{\widehat{R}_p}(x,2s)} \varphi^p \, d\Gamma\langle\Phi_\varepsilon(u)\rangle \\
&\overset{\text{(CL2)}}{=} \frac{1}{p-1}\int_{B_{\widehat{R}_p}(x,2s)} \varphi^p \, d\Gamma\langle u_\varepsilon; u_\varepsilon^{1-p}\rangle \\
&\overset{\text{(CL2)}}{=} \frac{1}{1-p}\left(\int_{B_{\widehat{R}_p}(x,2s)} d\Gamma\langle u_\varepsilon; \varphi^p u_\varepsilon^{1-p}\rangle - \int_{B_{\widehat{R}_p}(x,2s)} u_\varepsilon^{1-p} \, d\Gamma\langle u_\varepsilon; \varphi^p\rangle\right) \\
&\overset{(*)}{\leq} \frac{1}{1-p}\left(\mathcal{E}(u_\varepsilon; \varphi^p u_\varepsilon^{1-p}) - \int_{B_{\widehat{R}_p}(x,2s)} u_\varepsilon^{1-p} \, d\Gamma\langle u_\varepsilon; \varphi^p\rangle\right) \\
&\overset{(**)}{\leq} \frac{-1}{1-p}\int_{B_{\widehat{R}_p}(x,2s)} u_\varepsilon^{1-p} \, d\Gamma\langle u_\varepsilon; \varphi^p\rangle \\
&\overset{\text{(CL2)}}{=} \frac{p}{p-1}\int_{B_{\widehat{R}_p}(x,2s)} \varphi^{p-1} \, d\Gamma\langle\Phi_\varepsilon(u); \varphi\rangle \\
&\overset{(4.13)}{\leq} \frac{p}{p-1}\left(\frac{1}{2}\int_{B_{\widehat{R}_p}(x,2s)} \varphi^p \, d\Gamma\langle\Phi_\varepsilon(u)\rangle\right)^{(p-1)/p}\left(2^{p-1}\int_{B_{\widehat{R}_p}(x,2s)} d\Gamma\langle\varphi\rangle\right)^{1/p} \\
&\leq \frac{p}{p-1}\left(\frac{p-1}{2p}\int_{B_{\widehat{R}_p}(x,2s)} \varphi^p \, d\Gamma\langle\Phi_\varepsilon(u)\rangle + \frac{2^{p-1}}{p}\int_{B_{\widehat{R}_p}(x,2s)} d\Gamma\langle\varphi\rangle\right),
\end{aligned}$$

where we used Theorem 4.17 and $\Gamma\langle u_\varepsilon\rangle(X) = \mathcal{E}(u_\varepsilon)$ in $(*)$, the fact that u_ε is $\mathcal{E}$-superharmonic in $B_{\widehat{R}_p}(x,2s)$ in $(**)$, and Young's inequality in the last inequality. Hence we obtain $\int_{B_{\widehat{R}_p}(x,s)} d\Gamma\langle\Phi_\varepsilon(u)\rangle \leq p^{-1}2^p\mathcal{E}(\varphi)$. □

With these preparations, we can show the desired elliptic Harnack inequality as follows.

Theorem 6.37 (Elliptic Harnack inequality)**.** *Assume that there exist a Borel measure μ on X, $\beta, Q \in (0,\infty)$ with $\beta > Q$ and $A, C \in [1,\infty)$ such that the following conditions are satisfied.*

(i) *μ satisfies (6.34) and (6.35) holds for any $(x,s) \in X \times (0,\infty)$ and any $u \in \mathcal{F}$.*

(ii) *For any $(x,s) \in X \times (0,\infty)$ with $B_{\widehat{R}_p}(x,s) \neq X$,*

$$\inf\bigl\{\mathcal{E}(\varphi) \bigm| \varphi \in \mathcal{F}, \varphi|_{B_{\widehat{R}_p}(x,s)} = 1, \mathrm{supp}_X[\varphi] \subseteq B_{\widehat{R}_p}(x,2s)\bigr\} \leq C\frac{\mu\bigl(B_{\widehat{R}_p}(x,s)\bigr)}{s^\beta}. \tag{6.40}$$

(iii) $\{\Gamma\langle u\rangle\}_{u\in\mathcal{F}}$ *satisfies the chain rule* (CL2)*.*

Then there exist $C_{\mathrm{H}} \in (0,\infty)$ *and* $\delta_{\mathrm{H}} \in (0,1)$ *such that the following hold. Let* $(x,s) \in X \times (0,\infty)$ *with* $B_{\widehat{R}_p}(x,\delta_{\mathrm{H}}^{-1}s) \neq X$ *and let* $u \in \mathcal{F}$ *with* $u \geq 0$*. If* $\Gamma\langle u\rangle(X) = \mathcal{E}(u)$ *and* u *is* $\mathcal{E}$*-superharmonic in* $B_{\widehat{R}_p}(x,\delta_{\mathrm{H}}^{-1}s)$*, then*

$$\sup_{B_{\widehat{R}_p}(x,s)} u \leq C_{\mathrm{H}} \inf_{B_{\widehat{R}_p}(x,s)} u. \tag{6.41}$$

Proof. Let $\varepsilon > 0$ and $\delta_{\mathrm{H}} \coloneqq (2\widetilde{A})^{-1}$, where $\widetilde{A}$ is the constant in Lemma 6.35. Set $u_\varepsilon \coloneqq u+\varepsilon$, $M_\varepsilon \coloneqq \sup_{B_{\widehat{R}_p}(x,s)} u_\varepsilon$ and $m_\varepsilon \coloneqq \inf_{B_{\widehat{R}_p}(x,s)} u_\varepsilon$. By combining (6.36), (6.39) and (6.40), there exists $C_0 \in (0,\infty)$ independent of x,s,u,ε such that

$$\sup_{B_{\widehat{R}_p}(x,s)} \log u_\varepsilon - \inf_{B_{\widehat{R}_p}(x,s)} \log u_\varepsilon \leq C_0,$$

whence $\log\left(\frac{M_\varepsilon}{m_\varepsilon}\right) \leq C_0$. In particular, $M_\varepsilon/m_\varepsilon \leq e^{C_0}$. We obtain (6.41) by letting $\varepsilon \downarrow 0$. □

7 Self-similar p-resistance forms and p-energy measures

In this section, we investigate p-resistance forms by focusing on the self-similar case as in Section 5. Throughout this section, we fix $p \in (1,\infty)$ and a self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ with $S \geq 2$ and assume that K is connected.

7.1 Self-similar p-resistance forms

We first introduce the notion of *self-similar p-resistance form*. Note that the topology induced the p-resistance metric may be different from the original topology of K. We always equip K with its original topology.

Definition 7.1 (Self-similar p-resistance form)**.** Let $\rho = (\rho_i)_{i\in S} \in (0,\infty)^S$ and let $(\mathcal{E},\mathcal{F})$ be a p-resistance form on K. We say that $(\mathcal{E},\mathcal{F})$ is a *self-similar p-resistance form on* $\mathcal{L}$ *with weight* ρ if and only if $\mathcal{F} \subseteq C(K)$ and $(\mathcal{E},\mathcal{F})$ satisfies (5.5) and (5.6).

In the rest of this section, we also fix a self-similar p-resistance form $(\mathcal{E},\mathcal{F})$ on $\mathcal{L}$ with weight $\rho = (\rho_i)_{i\in S} \in (0,\infty)^S$.

The following properties of the p-resistance metric are elementary.

Proposition 7.2. (1) *For any* $x, y \in K$*,*

$$R_{\mathcal{E}}(F_w(x), F_w(y)) \leq \rho_w^{-1} R_{\mathcal{E}}(x,y) \tag{7.1}$$

(2) *If* $\min_{i\in S} \rho_i > 1$ *and* $\operatorname{diam}(K, \widehat{R}_{p,\mathcal{E}}) < \infty$*, then* $\widehat{R}_{p,\mathcal{E}}$ *is compatible with the original topology of* K*. In particular,* V_* *is dense in* $(K, \widehat{R}_{p,\mathcal{E}})$*.*

(3) *If* $\min_{i\in S}\rho_i > 1$ *and* $\mathcal{L}$ *is a p.-c.f. self-similar structure, then* $\widehat{R}_{p,\mathcal{E}}$ *is compatible with the original topology of* K*. In particular,* V_* *is dense in* $(K, \widehat{R}_{p,\mathcal{E}})$*.*

Remark 7.3. In the case $p = 2$, $\min_{i\in S}\rho_i > 1$ and $\mathcal{L}$ is a p.-c.f. self-similar structure, then it is known that there exists $c \in (0,\infty)$ such that for any $x, y \in K$ and any $w \in W_*$,

$$R_{\mathcal{E}}(F_w(x), F_w(y)) \geq c\rho_w^{-1}R_{\mathcal{E}}(x,y); \tag{7.2}$$

see [Kig03, Theorem A.1]. Such a result is also true for p-resistance form. See Theorem B.10.

Proof. (1): It is immediate from (5.6). (See [Kig01, Lemma 3.3.5] for the case $p = 2$.)

(2): We can follow the argument in [Kig09, Proof of Proposition B.2] to show that $\widehat{R}_{p,\mathcal{E}}$ is compatible with the original topology of K. (Note that the condition that $\mathcal{F}$ is dense in $C(K)$ in [Kig09, (RFA3)] is not used in [Kig09, Proposition B.2].) Then V_* is dense in $(K, \widehat{R}_{p,\mathcal{E}})$ since $\overline{V_*}^K = K$ by [Kig01, Lemma 1.3.11].

(3): We can follow the argument in [Kig01, Proof of Theorem 3.3.4]; see also Lemma 8.41. □

The following proposition presents compatible sequences of p-resistance forms having a self-similarity.

Proposition 7.4. *Assume that* $\widehat{R}_{p,\mathcal{E}}$ *is compatible with the original topology of* K*. Let* $n \in \mathbb{N}\cup\{0\}$ *and let* Λ *be a partition of* Σ*. Define* $V_{n,\Lambda} \coloneqq \bigcup_{w\in\Lambda}F_w(V_n)$*. Then for any* $u \in \mathcal{F}|_{V_{n,\Lambda}}$*,*

$$\mathcal{E}|_{V_{n,\Lambda}}(u) = \sum_{w\in\Lambda}\rho_w\mathcal{E}|_{V_n}(u\circ F_w). \tag{7.3}$$

Moreover, for any $w \in \Lambda$*,*

$$h^{\mathcal{E}}_{V_{n,\Lambda}}(u)\circ F_w = h^{\mathcal{E}}_{V_n}(u\circ F_w). \tag{7.4}$$

In particular, for any $m \in \mathbb{N}\cup\{0\}$ *and any* $u \in \mathcal{F}|_{V_{n+m}}$*,*

$$\mathcal{E}|_{V_{n+m}}(u) = \sum_{w\in W_m}\rho_w\mathcal{E}|_{V_n}(u\circ F_w). \tag{7.5}$$

Proof. Note that (7.5) follows from (7.3) by choosing $\Lambda = W_m$ and that $\mathcal{S} \coloneqq \bigl\{(V_{n,\Lambda}, \mathcal{E}_{V_{n,\Lambda}})\bigr\}_{n\in\mathbb{N}\cup\{0\}}$ is a compatible sequence of p-resistance forms by Proposition 6.15. Let $u \in \mathcal{F}|_{V_{n,\Lambda}}$. Then we see that

$$\begin{aligned}
\mathcal{E}|_{V_{n,\Lambda}}(u) &= \min\bigl\{\mathcal{E}(v) \bigm| v \in \mathcal{F} \text{ with } v|_{V_{n,\Lambda}} = u\bigr\} \\
&\overset{(5.7)}{=} \min\left\{\sum_{w\in\Lambda}\rho_w\mathcal{E}(v\circ F_w) \,\middle|\, v \in \mathcal{F} \text{ with } v|_{V_{n,\Lambda}} = u\right\} \\
&\geq \sum_{w\in\Lambda}\rho_w \min\bigl\{\mathcal{E}(v) \bigm| v \in \mathcal{F} \text{ with } v|_{V_n} = u\circ F_w\bigr\} = \sum_{w\in\Lambda}\rho_w\mathcal{E}|_{V_n}(u\circ F_w).
\end{aligned}$$

To prove the converse, we define $v \in C(K) = C(K, \widehat{R}_{p,\mathcal{E}})$ so that $v \circ F_w = h^{\mathcal{E}}_{V_n}[u \circ F_w]$ for any $w \in \Lambda$; note that such v is well-defined by (5.2). Then $v|_{V_{n,\Lambda}} = u$ and $v \in \mathcal{F}_{\mathcal{S}}$ by (5.5). Since

$$\mathcal{E}|_{V_{n,\Lambda}}(u) \le \mathcal{E}(v) \overset{(5.7)}{=} \sum_{w\in\Lambda} \rho_w \mathcal{E}(v \circ F_w) = \sum_{w\in\Lambda} \rho_w \mathcal{E}\bigl(h^{\mathcal{E}}_{V_n}[u \circ F_w]\bigr) = \sum_{w\in\Lambda} \rho_w \mathcal{E}|_{V_n}(u \circ F_w),$$

we have (7.3). Next we prove (7.4). We have $\mathcal{E}\bigl(h^{\mathcal{E}}_{V_{n,\Lambda}}[u] \circ F_w\bigr) \ge \mathcal{E}\bigl(h^{\mathcal{E}}_{V_n}[u \circ F_w]\bigr)$ for any $w \in \Lambda$. Since

$$\begin{aligned}\mathcal{E}|_{V_{n,\Lambda}}(u) = \mathcal{E}\bigl(h^{\mathcal{E}}_{V_{n,\Lambda}}[u]\bigr) &= \sum_{w\in\Lambda} \rho_w \mathcal{E}\bigl(h^{\mathcal{E}}_{V_{n,\Lambda}}[u] \circ F_w\bigr)\\ &\ge \sum_{w\in\Lambda} \rho_w \mathcal{E}\bigl(h^{\mathcal{E}}_{V_n}[u \circ F_w]\bigr) = \sum_{w\in\Lambda} \rho_w \mathcal{E}|_{V_n}(u \circ F_w) = \mathcal{E}|_{V_{n,\Lambda}}(u),\end{aligned}$$

we obtain $\mathcal{E}\bigl(h^{\mathcal{E}}_{V_{n,\Lambda}}[u] \circ F_w\bigr) = \mathcal{E}\bigl(h^{\mathcal{E}}_{V_n}[u \circ F_w]\bigr)$ for any $w \in \Lambda$. The uniqueness in Theorem 6.13 implies $h^{\mathcal{E}}_{V_{n,\Lambda}}[u] \circ F_w = h^{\mathcal{E}}_{V_n}[u \circ F_w]$. □

The following corollary is an immediate consequence of Proposition 6.19.

Corollary 7.5. *Assume that $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is a p.-c.f. self-similar structure and that $\widehat{R}_{p,\mathcal{E}}$ is compatible with the original topology of K. Then*

$$\mathcal{F} = \Bigl\{u \in C(K) \Bigm| \lim_{n\to\infty} \mathcal{E}|_{V_n}(u|_{V_n}) < \infty\Bigr\}. \tag{7.6}$$

$$\mathcal{E}(u; v) = \lim_{n\to\infty} \mathcal{E}|_{V_n}(u|_{V_n}; v|_{V_n}) \quad \text{for any } u, v \in \mathcal{F}. \tag{7.7}$$

The following proposition gives characterizations of $\mathcal{E}$-harmonic functions on $K \setminus V_n$.

Proposition 7.6. *Assume that $\widehat{R}_{p,\mathcal{E}}$ is compatible with the original topology of K. Let $n \in \mathbb{N} \cup \{0\}$. Then for each $h \in C\bigl(K, \widehat{R}_{p,\mathcal{E}}\bigr)$, the following two conditions are equivalent to each other:*

(1) *$h \in \mathcal{H}_{\mathcal{E},V_n}$.*

(2) *$h \circ F_w \in \mathcal{H}_{\mathcal{E},V_0}$ for any $w \in W_n$.*

If in addition $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is a p.-c.f. self-similar structure, then (1) (or (2)) is also equivalent to the following condition:

(3) *For any $m \in \mathbb{N}$ with $m > n$ and any $x \in V_m \setminus V_n$,*

$$\sum_{w\in W_m;\, x\in F_w(V_0)} \rho_w \mathcal{E}|_{V_0}\Bigl(h \circ F_w|_{V_0}; \mathbb{1}^{V_0}_{F_w^{-1}(x)}\Bigr) = 0. \tag{7.8}$$

Proof. To see (1) ⇒ (2), let us fix $w \in W_n$ and let $\varphi \in \mathcal{F}^0(K \setminus V_0)$. Then $(F_w)_*\varphi \in \mathcal{F}$ by (5.5) and $(F_w)_*\varphi \in \mathcal{F}^0(K \setminus V_n)$ by (5.2). By (5.6), we have

$$0 = \mathcal{E}(h; (F_w)_*\varphi) = \rho_w \mathcal{E}(h \circ F_w; \varphi),$$

which implies $h \circ F_w \in \mathcal{H}_{\mathcal{E},V_0}$. The converse implication (2) $\Rightarrow$ (1) is obvious from (5.6).

Next we prove the equivalence of (1) and (3) for a p.-c.f. self-similar structure $\mathcal{L}$. We first show (1) $\Rightarrow$ (3). For any $m > n$ and any $\varphi \in \mathcal{F}^0(K \setminus V_n)$, we note that $h^{\mathcal{E}}_{V_m}[\varphi|_{V_m}]\big|_{V_n} = 0$. Then, for any $h \in \mathcal{H}_{\mathcal{E},V_n}$, we have from (7.5) that

$$0 = \mathcal{E}|_{V_m}(h|_{V_m};\varphi|_{V_m}) = \sum_{w\in W_m} \rho_w \mathcal{E}|_{V_0}\bigl(h \circ F_w|_{V_0};\varphi \circ F_w|_{V_0}\bigr) \quad \text{for any } \varphi \in \mathcal{F}^0(K \setminus V_0).$$

By choosing $\varphi \in \mathcal{F}^0(K \setminus V_n)$ so that $\varphi|_{V_m} = \mathbb{1}^{V_m}_x$ for $x \in V_m \setminus V_n$, we obtain (3). We next suppoes that $h \in C(K)$ satisfies (7.8) and fix $\varphi \in \mathcal{F}^0(K \setminus V_n)$ in order to show the converse implication (3) $\Rightarrow$ (1). For $m > n$, we see from (7.5), $\varphi|_{V_n} = 0$ and (7.8) that

$$\begin{aligned}
\mathcal{E}|_{V_m}(h|_{V_m};\varphi|_{V_m}) &= \sum_{w\in W_m} \rho_w \mathcal{E}|_{V_0}\bigl(h \circ F_w|_{V_0};\varphi \circ F_w|_{V_0}\bigr)\\
&= \sum_{w\in W_m}\sum_{y\in V_0} \varphi(F_w(y))\rho_w \mathcal{E}|_{V_0}\bigl(h \circ F_w|_{V_0};\mathbb{1}^{V_0}_y\bigr)\\
&= \sum_{x\in V_m\setminus V_n} \varphi(x) \sum_{w\in W_m; x\in F_w(V_0)} \rho_w \mathcal{E}|_{V_0}\Bigl(h \circ F_w|_{V_0};\mathbb{1}^{V_0}_{F_w^{-1}(x)}\Bigr) = 0.
\end{aligned}$$

By letting $m \to \infty$, we obtain $\mathcal{E}(h;\varphi) = 0$ and hence $h \in \mathcal{H}_{\mathcal{E},V_n}$. □

Thanks to the self-similarity, we can get the following localized version of the weak comparison principle (Proposition 6.26).

Proposition 7.7 (A localized weak comparison principle)**.** *Assume that $\widehat{R}_{p,\mathcal{E}}$ is compatible with the original topology of K. Let $n \in \mathbb{N} \cup \{0\}$, $w \in W_n$, and let $u, v \in \mathcal{H}_{\mathcal{E},V_n}$ satisfy $u(x) \le v(x)$ for any $x \in F_w(V_0)$. Then $u(x) \le v(x)$ for any $x \in K_w$.*

Proof. This is immediate from a combination of Proposition 6.26 and the implication from (1) to (2) in Proposition 7.6. □

Next we will show a new monotonicity on the equal weight of the p-resistance form on a p.-c.f. self-similar structure in p. We need the following basic result, which is immediate from (5.2) and Proposition 2.9-(a).

Proposition 7.8. *Let $k \in \mathbb{N} \cup \{0\}$ and let E be a p-resistance form on V_k. Define $\mathcal{S}_\rho(E)\colon \mathbb{R}^{V_{k+1}} \to [0,\infty)$ by*

$$\mathcal{S}_\rho(E)(u) \coloneqq \sum_{i\in S} \rho_i E(u \circ F_i), \quad u \in \mathbb{R}^{V_k}. \tag{7.9}$$

Then $\mathcal{S}_\rho(E)$ is a p-resistance form on V_{k+1}.

The following theorem states the desired monotonicity. (See also Theorem 8.31 for a similar result in another framework including the generalized Sierpiński carpets.)

Theorem 7.9. *Let $p, q \in (1,\infty)$ with $p \le q$ and let $\rho_s \in (1,\infty)$ for each $s \in \{p,q\}$. Assume that K is connected, that $\mathcal{L}$ is a p.-c.f. self-similar structure and that $(\mathcal{E}_s, \mathcal{F}_s)$ is a self-similar s-resistance form on $\mathcal{L}$ with weight $(\rho_s)_{i\in S}$ for each $s \in \{p,q\}$. Then*

$$\rho_p^{1/(p-1)} \le \rho_q^{1/(q-1)}. \tag{7.10}$$

Proof. We start by some preparations on discrete energies. Let $s \in \{p,q\}$. For any s-resistance form E_s on V_0 and $n \in \mathbb{N}$, we define $\mathcal{S}_{\rho_s,n}(E_s)\colon \mathbb{R}^{V_n} \to [0,\infty)$ by

$$\mathcal{S}_{\rho_s,n}(E_s)(u) \coloneqq \rho_s^n \sum_{v\in W_n} E_s(u \circ F_v), \quad u \in \mathbb{R}^{V_n}.$$

Note that $\mathcal{S}_{\rho_s,1}^n = \mathcal{S}_{\rho_s,n}$ and that $\mathcal{S}_{\rho_s,n}(E)$ is also a s-resistance form on V_n by Proposition 7.8. We also define a s-resistance form $E_{s,n}$ on V_n by

$$E_{s,n}(u) \coloneqq \rho_s^n \sum_{v\in W_n} \sum_{\{x,y\}\in V_0} |u(F_v(x)) - u(F_v(y))|^s, \quad u \in \mathbb{R}^{V_n}.$$

Then $\mathcal{S}_{\rho_s,n}(E_{s,0}) = E_{s,n}$. Since both $E_{s,0}(\,\cdot\,)^{1/s}$ and $\mathcal{E}_s|_{V_0}(\,\cdot\,)^{1/s}$ are norms on the finite-dimensional vector space $\mathbb{R}^{V_0}/\mathbb{R}\mathbb{1}_{V_0}$, there exists a constant $C_s \ge 1$ depending only on s and $\#V_0$ such that

$$C_s^{-1} E_{s,0}(u) \le \mathcal{E}_s|_{V_0}(u) \le C_s E_{s,0}(u) \quad \text{for any } u \in \mathbb{R}^{V_0}. \tag{7.11}$$

Since $\mathcal{S}_{\rho_s,n}(\mathcal{E}_s|_{V_0}) = \mathcal{E}_s|_{V_n}$ by (7.5), we have from (7.11) that

$$C_s^{-1} E_{s,n}(u) \le \mathcal{E}_s|_{V_n}(u) \le C_s E_{s,n}(u) \quad \text{for any } n \in \mathbb{N}\cup\{0\} \text{ and any } u \in \mathbb{R}^{V_n}. \tag{7.12}$$

Now we move to the proof of (7.10). Let us fix $x_0, y_0 \in V_0$ with $x_0 \neq y_0$ and set $B \coloneqq \{x_0, y_0\}$. Then we can find $w \in W_*$ so that $B \cap K_w = \emptyset$ and $h_{p,w} \coloneqq h_p \circ F_w \notin \mathbb{R}\mathbb{1}_K$, where $h_p \coloneqq h_{V_0}^{\mathcal{E}_p}[\mathbb{1}_{x_0}]$. (If $h_p \circ F_w \in \mathbb{R}\mathbb{1}_K$ for any $w \in W_*$ with $B \cap K_w = \emptyset$, then we can easily obtain a contradiction by using (6.3) and [Kig01, Theorem 1.6.2], where the connectedness of K is used.) Since $c \coloneqq \inf_{x\in K_w} R_{\mathcal{E}_p}(x,B) > 0$ and $0 \le h_p \le 1$ by (6.29), for any $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathcal{E}_q|_{V_n}(h_{p,w}|_{V_n}) \\
&\overset{(7.12)}{\le} C_q E_{q,n}(h_{p,w}|_{V_n}) \\
&= C_q \rho_q^n \sum_{v\in W_n} \sum_{\{x,y\}\in E_0} |h_p(F_{wv}(x)) - h_p(F_{wv}(y))|^{q-p} \cdot |h_{p,w}(F_v(x)) - h_{p,w}(F_v(y))|^p \\
&\overset{(6.31)}{\le} C_q \rho_q^n \sum_{w\in W_n} \sum_{\{x,y\}\in E_0} \left(\frac{R_{\mathcal{E}_p}(F_{wv}(x), F_{wv}(y))}{R_{\mathcal{E}_p}(F_{wv}(x), B)} \right)^{\frac{q-p}{p-1}} \cdot |h_{p,w}(F_w(x)) - h_{p,w}(F_w(y))|^p \\
&\overset{(7.1)}{\le} \left(C_q c^{-(q-p)/(p-1)} \sup_{x,y\in K} R_{\mathcal{E}_p}(x,y)^{(q-p)/(p-1)} \right) \left(\rho_q \rho_p^{-(q-1)/(p-1)} \right)^n E_{p,n}(h_{p,w}|_{V_n}) \\
&\overset{(7.12)}{\le} \left(C_p C_q c^{-(q-p)/(p-1)} \sup_{x,y\in K} R_{\mathcal{E}_p}(x,y)^{(q-p)/(p-1)} \right) \left(\rho_q \rho_p^{-(q-1)/(p-1)} \right)^n \mathcal{E}_p(h_{p,w}).
\end{aligned} \tag{7.13}$$

Since both $\mathcal{E}_p(h_{p,w})$ and $\mathcal{E}_q(h_{p,w})$ are not equal to 0, we conclude that $\rho_q \rho_p^{-(q-1)/(p-1)} \ge 1$ by letting $n \to \infty$ in (7.13). This proves the desired monotonicity (7.10). □

7.2 Associated self-similar p-energy measures and Poincaré inequality

In this subsection, we show a Poincaré type inequality in terms of self-similar p-energy measures under some geometric assumptions on the p-resistance metric.

Recall that we fix a self-similar p-resistance form $(\mathcal{E}, \mathcal{F})$ on $\mathcal{L}$ with weight $\rho = (\rho_i)_{i\in S} \in (0,\infty)^S$. We also let $\{\Gamma_{\mathcal{E}}\langle u\rangle\}_{u\in\mathcal{F}}$ be the associated p-energy measures defined in (5.11). In the following definition, we introduce natural *scales* $\{\Lambda_s\}_{s\in(0,1]}$ with respect to the p-resistance metric $\widehat{R}_{p,\mathcal{E}} =: \widehat{R}_p$. See [Kig09, Kig20] for further details on scales.

Definition 7.10. (1) We define $\Lambda_1^{\widehat{R}_p} := \{\emptyset\}$,

$$\Lambda_s^{\widehat{R}_p} := \left\{ w \;\middle|\; w = w_1 \dots w_n \in W_* \setminus \{\emptyset\}, (\rho_{w_1\dots w_{n-1}})^{-1/(p-1)} > s \geq \rho_w^{-1/(p-1)} \right\}$$

for each $s \in (0,1)$. (Note that $\{\Lambda_s^{\widehat{R}_p}\}_{s\in(0,1]}$ is the scale associated with the weight function $g(w) := \rho_w^{-1/(p-1)}$; see [Kig20, Definition 2.3.1].)

(2) For each $(s,x) \in (0,1] \times K$, we define $\Lambda_{s,0}^{\widehat{R}_p}(x) := \{w \in \Lambda_s^{\widehat{R}_p} \mid x \in K_w\}$ and $U_0^{\widehat{R}_p}(x,s) := \bigcup_{w\in\Lambda_{s,0}^{\widehat{R}_p}(x)} K_w$. Inductively, for $M \in \mathbb{N}$, define $\Lambda_{s,M}^{\widehat{R}_p}(x) := \{w \in \Lambda_s^{\widehat{R}_p} \mid K_w \cap U_{M-1}^{\widehat{R}_p}(x,s) \neq \emptyset\}$ and $U_M^{\widehat{R}_p}(x,s) := \bigcup_{w\in\Lambda_{s,M}^{\widehat{R}_p}(x)} K_w$.

It is easy to see that $\lim_{s\downarrow 0} \min\{|w| \mid w \in \Lambda_s^{\widehat{R}_p}\} = \infty$, that $\Lambda_s^{\widehat{R}_p}$ is a partition of Σ for any $s \in (0,1]$, and that $\Lambda_{s_1}^{\widehat{R}_p} \leq \Lambda_{s_2}^{\widehat{R}_p}$ for any $s_1, s_2 \in (0,1]$ with $s_1 \leq s_2$. By [Kig20, Proposition 2.3.7], for any $x \in K$ and any $M \in \mathbb{N}\cup\{0\}$, $\{U_M^{\widehat{R}_p}(x,s)\}_{s\in(0,1]}$ is non-decreasing in s and forms a fundamental system of neighborhoods of x in K.

The following lemma is standard; see, e.g., [BB, Lemma 4.17].

Lemma 7.11. *Let $q \in [1,\infty)$ and let $(Y,\mathcal{A},\mu)$ be a measure space. For any $f \in L^1(Y,\mu)$ and any $E \in \mathcal{A}$ with $\mu(A) \in (0,\infty)$,*

$$⨍_E \left| f - ⨍_E f\,d\mu \right|^q d\mu \leq 2^q \inf_{a\in\mathbb{R}} ⨍_E |f-a|^q\,d\mu. \tag{7.14}$$

Now we can present a Poincaré inequality in this context.

Proposition 7.12 ((p,p)-Poincaré inequality for self-similar p-resistance form)**.** *Assume that there exist $\alpha_1, \alpha_2 \in (0,\infty)$ such that for any $(x,s) \in K \times (0,1]$,*

$$B_{\widehat{R}_p}(x,\alpha_1 s) \subseteq U_{M_*}^{\widehat{R}_p}(x,s) \subseteq B_{\widehat{R}_p}(x,\alpha_2 s). \tag{7.15}$$

Let μ be a Borel probability measure on K satisfying the following condition:

$$\inf_{(x,s)\in K\times(\alpha_1,\infty)} \mu\bigl(B_{\widehat{R}_p}(x,s)\bigr) > 0. \tag{7.16}$$

Then there exist $C, A \in (0,\infty)$ with $A \geq 1$ such that for any $(x,s) \in K \times (0,\infty)$ and any $u \in \mathcal{F}_{\mathrm{loc}}(B_{\widehat{R}_p}(x, As))$,

$$\fint_{B_{\widehat{R}_p}(x,s)} \left| u(y) - \fint_{B_{\widehat{R}_p}(x,s)} u \, d\mu \right|^p \mu(dy) \leq C s^{p-1} \Gamma_{\mathcal{E}}\langle u \rangle (B_{\widehat{R}_p}(x, As)). \tag{7.17}$$

Proof. We can assume that $\alpha_1 \leq \alpha_2$ and $\alpha_1 \leq 1$ without loss of generality. Set $c_* \coloneqq \left(\inf_{(x,s)\in K\times(\alpha_1,\infty)} \mu\big(B_{\widehat{R}_p}(x,s)\big)\right)^{-1} \in (0,\infty)$ and $A \coloneqq \alpha_1^{-1}(\alpha_2 \vee \operatorname{diam}(K, \widehat{R}_p))$. We first consider the case $s \in (\alpha_1, \infty)$. In this case $B_{\widehat{R}_p}(x, As) = K$ and

$$\begin{aligned}
\fint_{B_{\widehat{R}_p}(x,s)} \left| f - \fint_{B_{\widehat{R}_p}(x,s)} f \, d\mu \right|^p d\mu &\overset{(7.14)}{\leq} 2^p \fint_{B_{\widehat{R}_p}(x,s)} \left| f - \fint_K f \, d\mu \right|^p d\mu \\
&\leq 2^p c_* \int_K \left| f - \fint_K f \, d\mu \right|^p d\mu \\
&\leq 2^p c_* \int_K \fint_K |u(y) - u(z)|^p \, \mu(dy)\mu(dz) \\
&\overset{(6.3)}{\leq} 2^p c_* \operatorname{diam}(K, \widehat{R}_p)^{p-1} \mathcal{E}(u) = C_1 \Gamma_{\mathcal{E}}\langle u \rangle (K),
\end{aligned}$$

where we used Hólder's inequality in the third inequality and set $C_1 \coloneqq 2^p c_* \operatorname{diam}(K, \widehat{R}_p)^{p-1}$. This shows (7.17).

Next let $s \in (0, \alpha_1]$. Let U be a relatively compact open subset of K such that $U \supseteq U_{M_*}^{\widehat{R}_p}(x, \alpha_1^{-1}s)$ and let $u^\# \in \mathcal{F}$ satisfy $u = u^\#$ on U. For any $y, z \in B_{\widehat{R}_p}(x,s)$, there exists $\{v(i)\}_{i=1}^{2M_*+1} \subseteq \Lambda_{\alpha_1^{-1}s, M_*}^{\widehat{R}_p}(x)$ such that $y \in K_{v(1)}$, $z \in K_{v(2M_*+1)}$ and $K_{v(i)} \cap K_{v(i+1)} \neq \emptyset$ for each $i \in \{1, 2, \dots, 2M_*\}$. Let us fix $x_i \in K_{v(i)} \cap K_{v(i+1)}$ and $q_i \in V_0$ so that $x_i = F_{v(i)}(q_i)$. We note that, for any $y', z' \in K_{v(i)}$,

$$\begin{aligned}
|u(y') - u(z')|^p &= \left| u(F_{v(i)}(F_{v(i)}^{-1}(y'))) - u(F_{v(i)}(F_{v(i)}^{-1}(z'))) \right|^p \\
&\leq R_{\mathcal{E}}(F_{v(i)}^{-1}(y'), F_{v(i)}^{-1}(z')) \mathcal{E}(u^\# \circ F_{v(i)}) \\
&\overset{(5.12)}{\leq} \operatorname{diam}(K, \widehat{R}_p)^{p-1} \rho_{v(i)}^{-1} \Gamma_{\mathcal{E}}\langle u^\# \rangle (K_{v(i)}) = \operatorname{diam}(K, \widehat{R}_p)^{p-1} \rho_{v(i)}^{-1} \Gamma_{\mathcal{E}}\langle u \rangle (K_{v(i)}).
\end{aligned}$$

Hence

$$\begin{aligned}
&|u(y) - u(z)|^p \\
&\leq (2M_* + 1)^{p-1} \left(|u(y) - u(x_1)|^p + \sum_{i=1}^{2M_*-1} |u(x_i) - u(x_{i+1})|^p + |u(x_{2M_*}) - u(z)|^p \right) \\
&\overset{(6.3)}{\leq} \big((2M_* + 1) \operatorname{diam}(K, \widehat{R}_p)\big)^{p-1} \sum_{i=1}^{2M_*+1} \rho_{v(i)}^{-1} \Gamma_{\mathcal{E}}\langle u \rangle (K_{v(i)}) \\
&\leq C_2 s^{p-1} \Gamma_{\mathcal{E}}\langle u \rangle \left(\bigcup_{i=1}^{2M_*+1} K_{v(i)} \right) \leq C_2 s^{p-1} \Gamma_{\mathcal{E}_p}\langle u \rangle (B_{\widehat{R}}(x, \alpha_1^{-1}\alpha_2 s)),
\end{aligned} \tag{7.18}$$

where $C_2 \coloneqq \bigl((2M_* + 1)\alpha_1^{-1} \operatorname{diam}(K, \widehat{R}_p)\bigr)^{p-1}$. Now we see that

$$\fint_{B_{\widehat{R}_p}(x,s)} \left| u(y) - \fint_{B_{\widehat{R}_p}(x,s)} u \, d\mu \right|^p \mu(dy)$$
$$\leq \fint_{B_{\widehat{R}_p}(x,s)} \fint_{B_{\widehat{R}_p}(x,s)} |u(y) - u(z)|^p \, \mu(dz) \, \mu(dy) \overset{(7.18)}{\leq} C_2 s^{p-1} \Gamma_{\mathcal{E}}\langle u \rangle (B_{\widehat{R}_p}(x, As)),$$

where we used Hölder's inequality in the first inequality. This completes the proof. □

8 Constructions of p-energy forms satisfying the generalized p-contraction property

In the preceding sections, we have established fundamental results on p-energy forms satisfying the generalized p-contraction property $(\mathrm{GC})_p$, in particular p-Clarkson's inequality $(\mathrm{Cla})_p$. In this section, we would like to describe how to get a good p-energy form satisfying these properties in a few settings inspired by [Kig23] and [CGQ22]. (See also [KS.a] for another approach toward such a construction.)

8.1 Construction of p-energy forms on p-conductively homogeneous compact metric spaces

In this subsection, we verify that p-energy forms on *p-conductively homogeneous* compact metric spaces constructed in [Kig23] satisfy $(\mathrm{GC})_p$. We mainly follow the notation and terminology of [Kig23] in this and the next subsections. We refer to [Kig23, Chapter 2] and [Kig20, Chapters 2 and 3] for further details.

Throughout this subsection, we fix a locally finite, non-directed infinite tree (T, E_T) in the usual sense (see [Kig23, Definition 2.1] for example), and fix a *root* $\phi \in T$ of T. (Here T is the set of vertices and E_T is the set of edges.) For any $w \in T \setminus \{\phi\}$, we use $\overline{\phi w}$ to denote the unique simple path in T from ϕ to w.

Definition 8.1 ([Kig23, Definition 2.2])**.** (1) For $w \in T$, define $\pi \colon T \to T$ by

$$\pi(w) \coloneqq \begin{cases} w_{n-1} & \text{if } w \neq \phi \text{ and } \overline{\phi w} = (w_0, \dots, w_n), \\ \phi & \text{if } w = \phi. \end{cases}$$

Set $S(w) \coloneqq \{v \in T \mid \pi(v) = w\} \setminus \{w\}$. Moreover, for $k \in \mathbb{N}$, we define $S^k(w)$ inductively as

$$S^{k+1}(w) = \bigcup_{v \in S(w)} S^k(v).$$

For $A \subseteq T$, define $S^k(A) \coloneqq \bigcup_{w \in A} S^k(A)$.

(2) For $w \in T$ and $n \in \mathbb{N} \cup \{0\}$, define $|w| := \min\{n \geq 0 \mid \pi^n(w) = \phi\}$ and $T_n := \{w \in T \mid |w| = n\}$.
(3) Define $\Sigma := \{(\omega_n)_{n\geq 0} \mid \omega_n \in T_n \text{ and } \omega_n = \pi(\omega_{n+1}) \text{ for all } n \in \mathbb{N} \cup \{0\}\}$. For $\omega = (\omega_n)_{n\geq 0} \in \Sigma$, we write $[\omega]_n$ for $\omega_n \in T_n$. For $w \in T$, define $\Sigma_w := \{(\omega_n)_{n\geq 0} \in \Sigma \mid \omega_{|w|} = w\}$. For $A \subseteq T$, define $\Sigma_A := \bigcup_{w\in A} \Sigma_w$.

Let us recall the definition of a partition parametrized by a rooted tree (see [Kig20, Definition 2.2.1] and [Sas23, Lemma 3.6]).

Definition 8.2 (Partition parametrized by a tree)**.** Let K be a compact metrizable topological space without isolated points. A family of non-empty compact subsets $\{K_w\}_{w\in T}$ of K is called a *partition of K parametrized by the rooted tree* (T, E_T, ϕ) if and only if it satisfies the following conditions:

(P1) $K_\phi = K$ and for any $w \in T$, $\#K_w \geq 2$ and $K_w = \bigcup_{v\in S(w)} K_v$.
(P2) For any $w \in \Sigma$, $\bigcap_{n\geq 0} K_{[\omega]_n}$ is a single point.

In the rest of this subsection, we fix a compact metrizable topological space without isolated points K, a locally finite rooted tree (T, E_T, ϕ) satisfying $\#\{v \in T \mid \{v, w\} \in E_T\} \geq 2$ for any $w \in T$, a partition $\{K_w\}_{w\in T}$ parametrized by (T, E_T, ϕ), a metric d on K with $\mathrm{diam}(K, d) = 1$, and a Borel probability measure m on K. Now we introduce a graph approximation $\{(T_n, E_n^*)\}_{n\in\mathbb{N}\cup\{0\}}$ of K (see [Kig23, Proposition 2.8 and Definition 2.5-(3)]).

Definition 8.3. For $n \in \mathbb{N} \cup \{0\}$ and $A \subseteq T_n$, define

$$E_n^* := \bigl\{\{v, w\} \bigm| v, w \in T_n, v \neq w, K_v \cap K_w \neq \emptyset\bigr\},$$

and $E_n^*(A) = \bigl\{\{v, w\} \in E_n^* \bigm| v, w \in A\bigr\}$. Let d_n be the graph distance of (T_n, E_n^*). For $M \in \mathbb{N} \cup \{0\}$ and $w \in T_n$, define

$$\Gamma_M(w) := \{v \in T_n \mid d_n(v, w) \leq M\} \quad \text{and} \quad U_M(x; n) := \bigcup_{w\in T_n; x\in K_w} \bigcup_{v\in\Gamma_M(w)} K_v.$$

To state geometric assumptions in [Kig23], we need the following definition (see [Kig20, Definitions 2.2.1 and 3.1.15].)

Definition 8.4. (1) The partition $\{K_w\}_{w\in T}$ is said to be *minimal* if and only if $K_w \setminus \bigcup_{v\in T_{|w|}\setminus\{w\}} \neq \emptyset$ for any $w \in T$.
(2) The partition $\{K_w\}_{w\in T}$ is said to be *uniformly finite* if and only if $\sup_{w\in T} \#\Gamma_1(w) < \infty$. We set $L_* := \sup_{w\in T} \#\Gamma_1(w)$.

We also recall the following standard notion on metric measure spaces (see, e.g., [Hei, Kig20, MT] for further background).

Definition 8.5. (1) The measure m is said to be *volume doubling* with respect to the metric d if and only if there exists $C_{\mathrm{D}} \in (0, \infty)$ such that

$$m(B_d(x, 2r)) \leq C_{\mathrm{D}}\, m(B_d(x, r)) \quad \text{for any } (x, r) \in K \times (0, \infty). \tag{8.1}$$

The constant C_{D} is called the doubling constant of m.

(2) Let $Q \in (0,\infty)$. The measure m is said to be *Q-Ahlfors regular* with respect to the metric d if and only if there exists $C_{\mathrm{AR}} \in [1,\infty)$ such that

$$C_{\mathrm{AR}}^{-1} r^{Q} \leq m(B_d(x,r)) \leq C_{\mathrm{AR}} r^{Q} \quad \text{for any } (x,r) \in K \times (0, \operatorname{diam}(K,d)). \tag{8.2}$$

The measure m is simply said to be *Ahlfors regular* (with respect to d) if there exists $Q \in (0,\infty)$ such that m is Q-Ahlfors regular. Also, the metric d is said to be Q-Ahlfors regular if there exists a Borel measure μ on K which is Q-Ahlfors regular with respect to d.

(3) A metric ρ on K is said to be *quasisymmetric* to d, $\rho \underset{\mathrm{QS}}{\sim} d$ for short, if and only if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that

$$\frac{\rho(x,b)}{\rho(x,a)} \leq \eta\left(\frac{d(x,b)}{d(x,a)}\right) \quad \text{for any } x,a,b \in K \text{ with } x \neq a.$$

(4) The *Ahlfors regular conformal dimension* of (K,d) is the value $\dim_{\mathrm{ARC}}(K,d)$ defined as

$$\dim_{\mathrm{ARC}}(K,d) := \inf\left\{ Q > 0 \,\middle|\, \begin{array}{l} \text{there exists a metric } \rho \text{ on } K \text{ such that} \\ \rho \underset{\mathrm{QS}}{\sim} d \text{ and } \rho \text{ is } Q\text{-Ahlfors regular} \end{array} \right\}.$$

If m is Ahlfors regular, then it is clearly volume doubling. It is well known that the existence of a Q-Ahlfors regular m on (K,d) implies that the Hausdorff dimension of (K,d) is Q.

Now we recall basic geometric conditions in [Kig23]. The conditions (1), (2) and (3) below are important to follow the rest of this paper.

Assumption 8.6 ([Kig23, Assumption 2.15])**.** Let $(K,\mathcal{O})$ be a connected compact metrizable space, $\{K_w\}_{w\in T}$ a partition parametrized by the rooted tree (T,ϕ), d a metric on K that is compatible with the topology $\mathcal{O}$ and $\operatorname{diam}(K,d) = 1$ and m a Borel probability measure on K. There exist $M_* \in \mathbb{N}$ and $r_* \in (0,1)$ such that the following conditions (1)–(5) hold.

(1) K_w is connected for any $w \in T$, $\{K_w\}_{w\in T}$ is minimal and uniformly finite, and $\inf_{m\geq 0}\min_{w\in T_m} \#S(w) \geq 2$.

(2) There exist $c_i \in (0,\infty)$, $i \in \{1,\dots,5\}$, such that the following conditions (2A)-(2C) are true.

(2A) For any $w \in T$,

$$c_1 r_*^{|w|} \leq \operatorname{diam}(K_w,d) \leq c_2 r_*^{|w|}. \tag{8.3}$$

(2B) For any $n \in \mathbb{N}$ and $x \in K$,

$$B_d(x, c_3 r_*^n) \subseteq U_{M_*}(x;n) \subseteq B_d(x, c_4 r_*^n). \tag{8.4}$$

(In [Kig20], the metric d is called *M_*-adapted* if the condition (8.4) holds.)

(2C) For any $n \in \mathbb{N}$ and $w \in T_n$, there exists $x_w \in K_w$ satisfying

$$K_w \supseteq B_d(x_w, c_5 r_*^n). \tag{8.5}$$

(3) There exist $m_1 \in \mathbb{N}$, $\gamma_1 \in (0,1)$ and $\gamma \in (0,1)$ such that

$$m(K_w) \geq \gamma m(K_{\pi(w)}) \quad \text{for any } w \in T, \tag{8.6}$$

and

$$m(K_v) \leq \gamma_1 m(K_w) \quad \text{for any } w \in T \text{ and } v \in S^{m_1}(w). \tag{8.7}$$

Furthermore, m is volume doubling with respect to d and

$$m(K_w) = \sum_{v \in S(w)} m(K_v) \quad \text{for any } w \in T. \tag{8.8}$$

(4) There exists $M_0 \geq M_*$ such that for any $w \in T$, $k \geq 1$ and $v \in S^k(w)$,

$$\Gamma_{M_*}(v) \cap S^k(w) \subseteq \left\{ v' \in T_{|v|} \,\middle|\, \begin{array}{l} \text{there exist } l \leq M_0 \text{ and } (v_0, \ldots, v_l) \in S^k(w)^{l+1} \\ \text{such that } (v_{j-1}, v_j) \in E^*_{|v|} \text{ for any } j \in \{1, \ldots, l\} \end{array} \right\}.$$

(5) For any $w \in T$, $\pi(\Gamma_{M_*+1}(w)) \subseteq \Gamma_{M_*}(\pi(w))$.

We record a simple consequence of (8.8) in the next proposition.

Proposition 8.7. *Assume that the Borel probability measure m satisfies* (8.8). *Then $m(K_v \cap K_w) = 0$ for any $v, w \in T$ with $v \neq w$ and $|v| = |w|$.*

Proof. Let $n \in \mathbb{N} \cup \{0\}$ and $v, w \in T_n$ such that $v \neq w$. Enumerate T_n as $\{z(1), z(2), \ldots, z(l_n)\}$ such that $z(1) = v$ and $z(2) = w$, where $l_n = \# T_n$. Inductively, we define $\widetilde{K}_{z(j)}$ by

$$\widetilde{K}_{z(1)} = K_{z(1)}$$

and

$$\widetilde{K}_{z(j+1)} = K_{z(j+1)} \setminus \left(\bigcup_{i=1}^{k} \widetilde{K}_{z(i)} \right).$$

Then $\{\widetilde{K}_{z(j)}\}_{j=1}^{l_n}$ is a disjoint family of Borel sets and $\bigcup_{j=1}^{l_n} \widetilde{K}_{z(j)} = K$. Therefore,

$$1 = m(K) = \sum_{j=1}^{l_n} m\Big(\widetilde{K}_{z(j)}\Big).$$

On the other hand,(8.8) implies that

$$1 = m(K_\phi) = \sum_{j=1}^{l_n} m\big(K_{z(j)}\big).$$

Therefore, we conclude that $m\big(K_{z(j)} \setminus \widetilde{K}_{z(j)}\big) = 0$ for any $j \in \{1, \ldots, l_n\}$. In particular,

$$0 = m\Big(K_{z(2)} \setminus \widetilde{K}_{z(2)}\Big) = m\Big(K_w \setminus \big(K_w \setminus (K_v \cap K_w)\big)\Big) = m(K_v \cap K_w),$$

which completes the proof. $\square$

Next we introduce conductance, neighbor disparity constants and the notion of p-conductive homogeneity in Definitions 8.10, 8.8 and 8.11, following [Kig23, Sections 2.2, 2.3 and 3.3]. We will state some definitions and statements below for any $p \in (0,\infty)$ or $p \in [1,\infty)$, but on each such occasion we will explicitly declare that we let $p \in (0,\infty)$ or $p \in [1,\infty)$. Our main interest lies in the case $p \in (1,\infty)$.

Definition 8.8 ([Kig23, Definitions 2.17 and 3.4])**.** Let $p \in (0,\infty)$, $n \in \mathbb{N} \cup \{0\}$ and $A \subseteq T_n$.

(1) Define $\mathcal{E}^n_{p,A} \colon \mathbb{R}^A \to [0,\infty)$ by

$$\mathcal{E}^n_{p,A}(f) \coloneqq \sum_{\{u,v\} \in E^*_n(A)} |f(u) - f(v)|^p, \quad f \in \mathbb{R}^A.$$

We write $\mathcal{E}^n_p(f)$ for $\mathcal{E}^n_{p,T_n}(f)$.

(2) For $A_0, A_1 \subseteq A$, define $\mathrm{cap}^n_p(A_0, A_1; A)$ by

$$\mathrm{cap}^n_p(A_0, A_1; A) \coloneqq \inf\bigl\{\mathcal{E}^n_{p,A}(f) \bigm| f \in \mathbb{R}^A, f|_{A_i} = i \text{ for } i \in \{0,1\}\bigr\}.$$

(3) (Conductance constant) For $A_1, A_2 \subseteq A$ and $k \in \mathbb{N} \cup \{0\}$, define

$$\mathcal{E}_{p,k}(A_1, A_2, A) \coloneqq \mathrm{cap}^{n+k}_p\bigl(S^k(A_1), S^k(A_2); S^k(A)\bigr).$$

For $M \in \mathbb{N}$, define $\mathcal{E}_{M,p,k} \coloneqq \sup_{w \in T} \mathcal{E}_{p,k}(\{w\}, T_{|w|} \setminus \Gamma_M(w), T_{|w|})$.

Let us recall the notion of *covering system*, which will be used to define neighbor disparity constants and the notion of conductive homogeneity.

Definition 8.9 ([Kig23, Definitions 2.26-(3) and 2.29])**.** Let $N_T, N_E \in \mathbb{N}$.

(1) Let $n \in \mathbb{N} \cup \{0\}$ and $A \subseteq T_n$. A collection $\{G_i\}_{i=1}^k$ with $G_i \subseteq T_n$ is called a *covering of* $(A, E^*_n(A))$ *with covering numbers* (N_T, N_E) if and only if $A = \bigcup_{i=1}^k G_k$, $\max_{x \in A} \#\{i \mid x \in G_i\} \leq N_T$ and for any $(u,v) \in E^*_n(A)$, there exists $l \leq N_E$ and $\{w(1), \dots, w(l+1)\} \subseteq A$ such that $w(1) = u$, $w(l+1) = v$ and $(w(i), w(i+1)) \in \bigcup_{j=1}^k E^*_n(G_j)$ for any $i \in \{1, \dots, l\}$.

(2) Let $\mathscr{J} \subseteq \bigcup_{n \in \mathbb{N} \cup \{0\}} \{A \mid A \subseteq T_n\}$. The collection $\mathscr{J}$ is called a *covering system with covering number* (N_T, N_E) if and only if the following conditions are satisfied:
   - (i) $\sup_{A \in \mathscr{J}} \#A < \infty$.
   - (ii) For any $w \in T$ and $k \in \mathbb{N}$, there exists a finite subset $\mathscr{N} \subseteq \mathscr{J} \cap T_{|w|+k}$ such that $\mathscr{N}$ is a covering of $\bigl(S^k(w), E^*_{|w|+k}(S^k(w))\bigr)$ with covering numbers (N_T, N_E).
   - (iii) For any $G \in \mathscr{J}$ and $k \in \mathbb{N} \cup \{0\}$, if $G \subseteq T_n$, then there exists a finite subset $\mathscr{N} \subseteq \mathscr{J} \cap T_{n+k}$ such that $\mathscr{N}$ is a covering of $\bigl(S^k(G), E^*_{n+k}(S^k(G))\bigr)$ with covering numbers (N_T, N_E).

   The collection $\mathscr{J}$ is simply said to be a *covering system* if and only if there exist $(N_T, N_E) \in \mathbb{N}^2$ such that $\mathscr{J}$ is a covering system with covering number (N_T, N_E).

Definition 8.10 ([Kig23, Definitions 2.26-(1),(2) and 2.29])**.** Let $p \in (0,\infty)$, $n \in \mathbb{N}$ and $A \subseteq T_n$.

(1) For $k \in \mathbb{N} \cup \{0\}$ and $f\colon T_{n+k} \to \mathbb{R}$, define $P_{n,k}f\colon T_n \to \mathbb{R}$ by

$$(P_{n,k}f)(w) := \frac{1}{\sum_{v \in S^k(w)} m(K_v)} \sum_{v \in S^k(w)} f(v)m(K_v), \quad w \in T_n.$$

(Note that $P_{n,k}f$ depends on the measure m.)

(2) (Neighbor disparity constant) For $k \in \mathbb{N} \cup \{0\}$, define

$$\sigma_{p,k}(A) := \sup_{f\colon S^k(A) \to \mathbb{R}} \frac{\mathcal{E}^n_{p,A}(P_{n,k}f)}{\mathcal{E}^{n+k}_{p,S^k(A)}(f)}.$$

(3) Let $\mathscr{J} \subseteq \bigcup_{n \ge 0}\{A \mid A \subseteq T_n\}$ be a covering system. Define

$$\sigma^{\mathscr{J}}_{p,k,n} := \max\{\sigma_{p,k}(A) \mid A \in \mathscr{J}, A \subseteq T_n\} \quad \text{and} \quad \sigma^{\mathscr{J}}_{p,k} := \sup_{n \in \mathbb{N} \cup \{0\}} \sigma^{\mathscr{J}}_{p,k,n}.$$

Definition 8.11 ([Kig23, Definition 3.4])**.** Let $p \in [1, \infty)$. The compact metric space K (with a partition $\{K_w\}_{w \in T}$ and a measure m) is said to be *p-conductively homogeneous* if and only if there exists a covering system $\mathscr{J}$ such that

$$\sup_{k \in \mathbb{N} \cup \{0\}} \sigma^{\mathscr{J}}_{p,k} \mathcal{E}_{M_*,p,k} < \infty. \tag{8.9}$$

When we would like to clarify which partition is considered, we also say that K is p-conductively homogeneous with respect to $\{K_w\}_{w \in T}$.

The next consequence of (8.9) is more important than the original definition of the p-conductive homogeneity for our purpose.

Theorem 8.12 (A part of [Kig23, Theorem 3.30])**.** *Let $p \in [1, \infty)$ and assume that Assumption 8.6 holds. If K is p-conductively homogeneous, then there exist $\alpha_0, \alpha_1 \in (0, \infty)$, $\sigma_p \in (0, \infty)$ and a covering system $\mathscr{J}$ such that for any $k \in \mathbb{N} \cup \{0\}$,*

$$\alpha_0 \sigma_p^{-k} \le \mathcal{E}_{M_*,p,k} \le \alpha_1 \sigma_p^{-k} \quad \textit{and} \quad \alpha_0 \sigma_p^{k} \le \sigma^{\mathscr{J}}_{p,k} \le \alpha_1 \sigma_p^{k}. \tag{8.10}$$

In particular, the constant σ_p is determined by the following limit:

$$\sigma_p = \lim_{k \to \infty} \left(\mathcal{E}_{M_*,p,k}\right)^{-1/k}. \tag{8.11}$$

Remark 8.13. The existence of the limit in (8.11) is true without the p-conductive homogeneity. Indeed, if $(K, d, \{K_w\}_{w \in T})$ satisfies the conditions Assumption 8.6-(1),(2),(4),(5), then [Kig23, Theorem 2.23] together with Fekete's lemma implies the existence of the limit in (8.11) for *any* $p \in (0, \infty)$. For convenience, we call σ_p the *p-scaling factor* of $(K, d, \{K_w\}_{w \in T})$.

We also recall the "Sobolev space" $\mathcal{W}^p$ introduced in [Kig23, Lemma 3.13].

Definition 8.14. Let $p \in [1,\infty)$. Assume that Assumption 8.6-(1),(2),(4),(5) hold and let σ_p be the constant in (8.11).

(1) For $n \in \mathbb{N} \cup \{0\}$, define $P_n \colon L^1(K,m) \to \mathbb{R}$ by $P_n f(w) := \fint_{K_w} f\,dm$, $w \in T_n$.

(2) Define $\mathcal{N}_p \colon L^p(K,m) \to [0,\infty]$ and a linear subspace $\mathcal{W}^p$ of $L^p(K,m)$ by

$$\mathcal{N}_p(f) := \left(\sup_{n \in \mathbb{N}\cup\{0\}} \sigma_p^n \mathcal{E}_p^n(P_n f)\right)^{1/p}, \quad f \in L^p(K,m),$$
$$\mathcal{W}^p := \bigl\{ f \in L^p(K,m) \bigm| \mathcal{N}_p(f) < \infty \bigr\},$$

and we equip $\mathcal{W}^p$ the norm $\|\cdot\|_{\mathcal{W}^p}$ defined by

$$\|f\|_{\mathcal{W}^p} := \Bigl(\|f\|_{L^p(K,m)}^p + \mathcal{N}_p(f)^p\Bigr)^{1/p}, \quad f \in \mathcal{W}^p.$$

(3) For a linear subspace $\mathcal{D}$ of $\mathcal{W}^p$, we define

$$\mathcal{U}_p(\mathcal{D}) := \left\{ \mathscr{E} \colon \mathcal{D} \to [0,\infty) \;\middle|\; \begin{array}{l} \mathscr{E}^{1/p} \text{ is a seminorm on } \mathcal{D}, \text{ there exist } \alpha_0, \alpha_1 \in (0,\infty) \\ \text{such that } \alpha_0 \mathcal{N}_p(f) \le \mathscr{E}(f)^{1/p} \le \alpha_1 \mathcal{N}_p(f) \text{ for any } f \in \mathcal{D} \end{array} \right\}.$$

For simplicity, set $\mathcal{U}_p := \mathcal{U}_p(\mathcal{W}^p)$.

(4) For $n \in \mathbb{N} \cup \{0\}$ and $A \subseteq T_n$, we define $\widetilde{\mathcal{E}}_{p,A}^n \colon L^p(K,m) \to [0,\infty)$ by

$$\widetilde{\mathcal{E}}_{p,A}^n(f) := \sigma_p^n \mathcal{E}_{p,A}^n(P_n f), \quad f \in L^p(K,m).$$

We also set $\widetilde{\mathcal{E}}_p^n(f) := \widetilde{\mathcal{E}}_{p,T_n}^n(f)$.

We have the following property on $\mathcal{N}_p$ thanks to the connectedness of K and Assumption 8.6-(3).

Proposition 8.15. *Let $p \in [1,\infty)$. Assume that Assumption 8.6 holds. Then $\mathcal{N}_p(f) = 0$ if and only if there exists $c \in \mathbb{R}$ such that $f(x) = c$ for m-a.e. $x \in K$.*

Proof. It is clear that $\mathcal{N}_p(f) = 0$ if f is constant. Suppose that $f \in L^p(K,m)$ satisfies $\mathcal{N}_p(f) = 0$. Note that (T_n, E_n^*) is a connected graph for each $n \in \mathbb{N} \cup \{0\}$ (see [Kig23, Proposition 2.8]). Therefore, $\mathcal{N}_p(f) = 0$ implies that there exists $c_n \in \mathbb{R}$ such that $P_n f(w) = c_n$ for any $n \in \mathbb{N} \cup \{0\}$ and any $w \in T_n$. By (8.8), we have $c_n = c_{n+1}$ and hence there exists $c \in \mathbb{R}$ such that $c_n = c$ for any $n \in \mathbb{N} \cup \{0\}$. Now we let $\mathscr{L}_f \subseteq K$ denote the set of *Lebesgue points of* f, i.e.,

$$\mathscr{L}_f := \left\{ x \in K \;\middle|\; \lim_{r \downarrow 0} \fint_{B_d(x,r)} |f(x) - f(y)|\, m(dx) = 0 \right\}. \tag{8.12}$$

Then, by the volume doubling property of m and the Lebesgue differentiation theorem (see, e.g., [Hei, Theorem 1.8]), we have $\mathscr{L}_f \in \mathcal{B}(K)$ and $m(K \setminus \mathscr{L}_f) = 0$. For any $x \in \mathscr{L}_f$ and any $n \in \mathbb{N} \cup \{0\}$, by Proposition 8.7 and Assumption 8.6-(2),(3),

$$|f(x) - c| = \left| f(x) - \fint_{U_{M_*}(x;n)} f\,dm \right| \le \frac{1}{m(U_{M_*}(x;n))} \int_{B_d(x, c_4 r_*^n)} |f(x) - f(y)|\, m(dy)$$

$$\leq C \fint_{B_d(x,c_4 r_*^n)} |f(x) - f(y)|\, m(dy),$$

where we used (8.4) and the volume doubling property of m in the last inequality, and $C \in (0,\infty)$ is independent of x, f and n. By letting $n \to \infty$ in the estimate above, we obtain $f(x) = c$ for any $x \in \mathscr{L}_f$, which completes the proof. □

As shown in [Shi24, Kig23], $\mathcal{W}^p$ is a nice Banach space embedded in $C(K)$ if K is p-conductively homogeneous and $p > \dim_{\mathrm{ARC}}$. In general, we can show the following theorem.

Theorem 8.16. *Let $p \in [1,\infty)$. Assume that $(K, d, \{K_w\}_{w\in T}, m)$ satisfies Assumption 8.6 and that K is p-conductively homogeneous. Then $\mathcal{W}^p$ is a Banach space and $\mathcal{W}^p$ is dense in $L^p(K,m)$. If $p \in (1,\infty)$, then $\mathcal{W}^p$ is reflexive and separable. Moreover, if in addition $p > \dim_{\mathrm{ARC}}(K,d)$, then $\mathcal{W}^p$ can be identified with a subspace of $C(K)$ and $\mathcal{W}^p$ is dense in $C(K)$ with respect to the uniform norm.*

Remark 8.17. By [Kig20, Theorem 4.6.9], the condition $p > \dim_{\mathrm{ARC}}(K,d)$ is equivalent to $\sigma_p > 1$.

Proof. Note that $\mathcal{W}^p$ is a Banach space by [Kig23, Lemma 3.24] and that $\mathcal{W}^p$ is dense in $L^p(K,m)$ by [Kig23, Lemma 3.28].

In the rest of this proof, we assume that $p \in (1,\infty)$. Let us show that $\mathcal{W}^p$ is reflexive. Theorem 8.12 and [Kig23, Lemma 2.27] together imply that there exists a constant $C \in (0,\infty)$ such that for any $k, l \in \mathbb{N}$, $A \subseteq T_k$ and $f \in \mathbb{R}^{S^l(A)}$,

$$\widetilde{\mathcal{E}}_{p,A}^{k}(P_{k,l}f) \leq C\widetilde{\mathcal{E}}_{p,S^l(A)}^{k+l}(f). \tag{8.13}$$

The rest of the proof is very similar to [MS23+, Proof of Theorem 6.17], so we give a sketch (see also [Shi24, Theorem 5.9] and the proof of Theorem 8.19-(a) below). Let $\|\cdot\|_{p,n} \coloneqq \left(\|\cdot\|_{L^p(K,m)}^p + \widetilde{\mathcal{E}}_p^n(\,\cdot\,)\right)^{1/p}$, which can be regarded as the L^p-norm on $K \sqcup E_n^*$. Also, we consider $\widetilde{\mathcal{E}}_p^n$ as a $[0,\infty]$-valued functional on $L^p(K,m)$. From [Dal, Theorem 8.5 and Proposition 11.6], by extracting a subsequence of $\{\widetilde{\mathcal{E}}_p^n\}_{n\in\mathbb{N}}$ if necessary, we can assume that $\{\widetilde{\mathcal{E}}_p^n\}_{n\in\mathbb{N}}$ Γ-converges to some p-homogeneous functional $E_p\colon L^p(K,m) \to [0,\infty]$ as $n \to \infty$. Then $\{\|\cdot\|_{p,n}\}_{nin\mathbb{N}}$ Γ-converges to $|||\cdot||| \coloneqq \left(\|\cdot\|_{L^p(K,m)}^p + E_p\right)^{1/p}$ as $n \to \infty$, and hence $(|||\cdot|||^p, \mathcal{W}^p)$ is a p-energy form on (K,m) satisfying $(\mathrm{Cla})_p$. By using (8.13) and noting that $\lim_{k\to\infty} P_n f_k(w) = P_n f(w)$ for any $n \in \mathbb{N}\cup\{0\}$, any $w \in T_n$ and any $f, f_k \in L^p(K,m)$ with $\lim_{k\to\infty} \|f - f_k\|_{L^p(K,m)} = 0$, we can show that $|||\cdot|||$ is a norm on $\mathcal{W}^p$ that is equivalent to $\|\cdot\|_{\mathcal{W}^p}$. Thus, $\mathcal{W}^p$ is reflexive by Proposition 3.4 and the Milman–Pettis theorem. The separability of $\mathcal{W}^p$ immediately follows from [AHM23, Proposition 4.1] since $L^p(K,m)$ is separable and the inclusion map of $\mathcal{W}^p$ into $L^p(K,m)$ is a continuous linear injection.

In the case $p > \dim_{\mathrm{ARC}}(K,d)$, $\mathcal{W}^p$ can be identified with a subspace of $C(K)$ and is dense in $(C(K), \|\cdot\|_{\sup})$ by [Kig23, Lemmas 3.15, 3.16 and 3.19]. □

Let us introduce an important value, p-walk dimension, which will be a main topic in Section 9.

Definition 8.18 (p-Walk dimension)**.** Let $p \in (0,\infty)$. Assume that $(K, d, \{K_w\}_{w\in T})$ satisfies Assumption 8.6-(1),(2),(4),(5). Let $r_* \in (0,1)$ be the constant in (8.4), let σ_p be the p-scaling factor of $(K, d, \{K_w\}_{w\in T})$ (see (8.11) and Remark 8.13). We define $\tau_p \in \mathbb{R}$ by

$$\tau_p \coloneqq \frac{\log \sigma_p}{\log r_*^{-1}}. \tag{8.14}$$

If in addition m is Ahlfors regular with respect to d, then we define $d_{\mathrm{w},p} \in \mathbb{R}$ by

$$d_{\mathrm{w},p} \coloneqq d_{\mathrm{f}} + \tau_p, \tag{8.15}$$

where d_{f} denotes the Hausdorff dimension of (K, d). We call $d_{\mathrm{w},p}$ the *p-walk dimension* of $(K, d, \{K_w\}_{w\in T})$.

Now we prove the main result in this subsection, which is an improvement of [Kig23, Theorem 3.21].

Theorem 8.19. *Let $p \in (1,\infty)$. Assume that $(K, d, \{K_w\}_{w\in T}, m)$ satisfies Assumption 8.6 and that K is p-conductively homogeneous. Then there exist $\widehat{\mathcal{E}}_p \colon \mathcal{W}^p \to [0,\infty)$ and $c \in (0,\infty)$ such that the following hold:*

(a) *$(\widehat{\mathcal{E}}_p)^{1/p}$ is a seminorm on $\mathcal{W}^p$ and*

$$c\mathcal{N}_p(f) \le \widehat{\mathcal{E}}_p(f)^{1/p} \le \mathcal{N}_p(f) \quad \text{for any } f \in \mathcal{W}^p. \tag{8.16}$$

(b) *$(\widehat{\mathcal{E}}_p, \mathcal{W}^p)$ is a p-energy form on (K,m) satisfying* (GC)$_p$.

(c) (Invariance) *Let $\mathsf{T} \colon (K, \mathcal{B}(K), m) \to (K, \mathcal{B}(K), m)$ be Borel measurable and preserve m, i.e., $\mathsf{T}^{-1}(A) \in \mathcal{B}(K)$ and $m(\mathsf{T}^{-1}(A)) = m(A)$ for any $A \in \mathcal{B}(K)$. Then $f \circ \mathsf{T} \in \mathcal{W}^p$ and $\widehat{\mathcal{E}}_p(f \circ \mathsf{T}) = \widehat{\mathcal{E}}_p(f)$ for any $f \in \mathcal{W}^p$.*

(d) *If in addition $p > \dim_{\mathrm{ARC}}(K, d)$, then $(\widehat{\mathcal{E}}_p, \mathcal{W}^p)$ is a regular p-resistance form on K and there exist $C \in [1,\infty)$ such that*

$$C^{-1} d(x,y)^{\tau_p} \le R_{\widehat{\mathcal{E}}_p}(x,y) \le C d(x,y)^{\tau_p} \quad \text{for any } x, y \in K. \tag{8.17}$$

Proof. The most part of the proof will be very similar to that in [Kig23, Theorem 3.21], but we present the details because we do not assume $p > \dim_{\mathrm{ARC}}(K, d)$ unlike [Kig23, Theorem 3.21]. Let $\widehat{\mathcal{E}}_p$ be a subsequential Γ-limit of $\{\widetilde{\mathcal{E}}_p^n\}_n$ with respect to the topology of $L^p(K,m)$ as in [Kig23, Proof of Theorem 3.21], i.e., there exists a subsequence $\{\widetilde{\mathcal{E}}_p^{n'}\}_{n'}$ Γ-converging to $\widehat{\mathcal{E}}_p$ with respect to $L^p(K,m)$ as $n' \to \infty$. (Note that such a subsequential Γ-limit exists by [Dal, Theorem 8.5].)

(a): $\widehat{\mathcal{E}}_p$ is p-homogeneous by [Dal, Proposition 11.6]. The triangle inequality for $\widehat{\mathcal{E}}_p(\,\cdot\,)^{1/p}$ will be included in the proof of (b), so we shall prove (8.16). From the definition of the Γ-convergence, it is immediate that $\widehat{\mathcal{E}}_p(f) \le \liminf_{n\to\infty} \widetilde{\mathcal{E}}_p^n(f) \le \mathcal{N}_p(f)^p$.

Let us show the former inequality in (8.16). Let $f \in \mathcal{W}^p$ and let $\{f_{n'}\}_{n'}$ be a recovery sequence of $\{\widetilde{\mathcal{E}}_p^n\}_{n'}$ at f, i.e., $\lim_{n'\to\infty} \|f - f_{n'}\|_{L^p(K,m)} = 0$ and $\widehat{\mathcal{E}}_p(f) = \lim_{n'\to\infty} \widetilde{\mathcal{E}}_p^{n'}(f_{n'})$. Since $\lim_{n'\to\infty} P_k f_{n'}(w) = P_k f(w)$ for any $k \in \mathbb{N}$ and any $w \in T_k$, by (8.13),

$$\widetilde{\mathcal{E}}_p^k(f) = \lim_{n'\to\infty} \widetilde{\mathcal{E}}_p^k(f_{n'}) \le C \lim_{n'\to\infty} \widetilde{\mathcal{E}}_p^{n'}(f_{n'}) = C\widehat{\mathcal{E}}_p(f),$$

where $C \in (0,\infty)$ is the constant in (8.13). We obtain the desired estimate by taking the supremum over $k \in \mathbb{N} \cup \{0\}$.

(b): Let us fix $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,p]$, $q_2 \in [p,\infty]$ and $T = (T_1,\dots,T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfying (2.1). Define $Q_n \colon L^1(K,m) \to L^1(K,m)$ by

$$Q_n f \coloneqq \sum_{w\in T_n} P_n f(w) \mathbb{1}_{K_w} \quad \text{for } f \in L^1(K,m). \tag{8.18}$$

Note that $\|Q_n\|_{L^p(K,m)\to L^p(K,m)} \le 1$ by (8.8) and Hölder's inequality. Let us show $\|f - Q_n f\|_{L^p(K,m)} \to 0$ as $n \to \infty$ for any $f \in L^p(K,m)$. Define the Hardy–Littlewood maximal operator $\mathscr{M} \colon L^p(K,m) \to L^0(K,m)$ by

$$\mathscr{M} f(x) = \sup_{r>0} \fint_{B_d(x,r)} |f(y)|\, m(dy), \quad x \in K.$$

Since m is volume doubling with respect to d by Assumption 8.6-(3), [HKST, Theorem 3.5.6] implies that there exists a constant $C \in (0,\infty)$ such that $\|\mathscr{M} f\|_{L^p(K,m)} \le C\|f\|_{L^p(K,m)}$ for any $f \in L^p(K,m)$. We also easily see that for any $f \in L^p(K,m)$ and any $x \in K$,

$$\begin{aligned} |Q_n f(x)| \le \sum_{w\in T_n; x\in K_w} |P_n f(w)| &\le \sum_{w\in T_n; x\in K_w} \frac{m(B_d(x,2c_2 r_*^n))}{m(K_w)} \fint_{B_d(x,2c_2 r_*^n)} |f|\, dm \\ &\le \sum_{w\in T_n; x\in K_w} \frac{m(B_d(x,2c_2 r_*^n))}{m(B_d(x_w,c_5 r_*^n))} \mathscr{M} f(x) \le C_1 \mathscr{M} f(x), \end{aligned}$$

where $x_w \in K_w$ and c_2, c_5 are the same as in Assumption 8.6-(2) and we used the volume doubling property in the last inequality, and $C_1 \in (0,\infty)$ is a constant depending only on $\sup_{w\in T} \#\Gamma_1(w)$, c_2, c_5 and the doubling constant of m. Let $f \in L^p(K,m)$ and let $\mathscr{L}_f \subseteq K$ denote the set of Lebesgue points of f (recall (8.12)). Then $\mathscr{L}_f \in \mathcal{B}(K)$ and $m(K \setminus \mathscr{L}_f) = 0$ by the Lebesgue differentiation theorem for a volume doubling metric measure space (see, e.g., [Hei, Theorem 1.8]). Since

$$\begin{aligned} |f(x) - Q_n f(x)| &\le \sum_{w\in T_n; x\in K_w} \fint_{K_w} |f(x) - f(y)|\, m(dy) \\ &\le C_1 \fint_{B_d(x,2c_2 r_*^n)} |f(x) - f(y)|\, m(dy), \end{aligned}$$

we have $|f(x) - Q_n f(x)| \to 0$ as $n \to \infty$ for any $x \in \mathscr{L}_f$. Now the dominated convergence theorem implies $\|f - Q_n f\|_{L^p(K,m)} \to 0$.

Let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in (\mathcal{W}^p)^{n_1}$ and choose a recovery sequence $\{u_{k,n'}\}_{n'}$ of $\{\widetilde{\mathcal{E}}_p^{n'}\}_{n'}$ at u_k for each $k \in \{1, \dots, n_1\}$. For brevity, we write $\boldsymbol{u}_{n'} = (u_{1,n'}, \dots, u_{n_1,n'})$ and

$$\begin{aligned} P_{n'}\boldsymbol{u}_{n'}(v) &= \bigl(P_{n'}u_{1,n'}(v), \dots, P_{n'}u_{n_1,n'}(v)\bigr) \in \mathbb{R}^{n_1}, \quad v \in T_{n'}, \\ Q_{n'}\boldsymbol{u}_{n'}(v) &= \bigl(Q_{n'}u_{1,n'}(v), \dots, Q_{n'}u_{n_1,n'}(v)\bigr) \in \mathbb{R}^{n_1}, \quad v \in T_{n'}. \end{aligned}$$

Note that $\|u_{n'} - Q_{n'}u_{k,n'}\|_{L^p(K,m)} \to 0$ as $n' \to \infty$ by the fact proved in the previous paragraph. Similar to an argument in [Kig23, p. 46], by using $\|Q_n\|_{L^p(K,m)\to L^p(K,m)} \le 1$ and the estimate (2.21), we have

$$\|T_l(\boldsymbol{u}) - T_l(Q_{n'}\boldsymbol{u}_{n'})\|_{L^p(K,m)} \xrightarrow[n'\to\infty]{} 0 \quad \text{for any } l \in \{1, \dots, n_2\}. \tag{8.19}$$

Also, we note that

$$P_{n'}\bigl(T_l(Q_{n'}\boldsymbol{u}_{n'})\bigr) = T_l(P_{n'}\boldsymbol{u}_{n'}) \in \mathbb{R}^{T_{n'}} \quad \text{for any } l \in \{1, \dots, n_2\}. \tag{8.20}$$

With these preparations, we prove (GC)$_p$ for $(\widehat{\mathcal{E}}_p, \mathcal{W}^p)$. We suppose that $q_2 < \infty$ since the case $q_2 = \infty$ is similar. By (8.19) and (8.20), we see that

$$\begin{aligned} \sum_{l=1}^{n_2} \widehat{\mathcal{E}}_p\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p} &\overset{(8.19)}{\le} \sum_{l=1}^{n_2} \liminf_{n'\to\infty} \widetilde{\mathcal{E}}_p^{n'}\bigl(T_l(Q_{n'}\boldsymbol{u}_{n'})\bigr)^{q_2/p} \\ &\overset{(8.20)}{\le} \liminf_{n'\to\infty} \sum_{l=1}^{n_2} \left[\frac{\sigma_p^{n'}}{2} \sum_{(v,w)\in E_{n'}^{*}} |T_l(P_{n'}\boldsymbol{u}_{n'}(v)) - T_l(P_{n'}\boldsymbol{u}_{n'}(w))|^{q_2\cdot\frac{p}{q_2}}\right]^{q_2/p} \\ &\overset{(2.19)}{\le} \liminf_{n'\to\infty} \left(\frac{\sigma_p^{n'}}{2} \sum_{(v,w)\in E_{n'}^{*}} \|T(P_{n'}\boldsymbol{u}_{n'}(v)) - T(P_{n'}\boldsymbol{u}_{n'}(v))\|_{\ell^{q_2}}^{p}\right)^{q_2/p} \\ &\overset{(2.1)}{\le} \liminf_{n'\to\infty} \left(\frac{\sigma_p^{n'}}{2} \sum_{(v,w)\in E_{n'}^{*}} \|P_{n'}\boldsymbol{u}(v) - P_{n'}\boldsymbol{u}(v)\|_{\ell^{q_1}}^{p}\right)^{q_2/p} \\ &\le \liminf_{n'\to\infty} \left(\frac{\sigma_p^{n'}}{2} \sum_{(v,w)\in E_{n'}^{*}} \left[\sum_{k=1}^{n_1} |P_{n'}u_{k,n'}(v) - P_{n'}u_{k,n'}(w)|^{p\cdot\frac{q_1}{p}}\right]^{p/q_1}\right)^{q_2/p} \\ &\overset{(*)}{\le} \liminf_{n'\to\infty} \left(\sum_{k=1}^{n_1} \left[\frac{\sigma_p^{n'}}{2} \sum_{(v,w)\in E_{n'}^{*}} |P_{n'}u_{k,n'}(v) - P_{n'}u_{k,n'}(w)|^{p}\right]^{q_1/p}\right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}} \\ &\le \left(\sum_{k=1}^{n_1} \limsup_{n'\to\infty} \widetilde{\mathcal{E}}_p^{n'}(u_{k,n'})^{q_1/p}\right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}} \le \left(\sum_{k=1}^{n_1} \widehat{\mathcal{E}}_p(u_k)^{q_1/p}\right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}}, \end{aligned} \tag{8.21}$$

where we used the triangle inequality for the ℓ^{p/q_1}-norm on E_n^{*} in $(*)$. Hence $(\widehat{\mathcal{E}}_p, \mathcal{W}^p)$ satisfies (GC)$_p$.

(c): This is clear from $P_n f = P_n(f \circ \mathsf{T}) \in \mathbb{R}^{T_n}$ for any $n \in \mathbb{N} \cup \{0\}$, $f \in L^p(K, m)$.

(d): In the case $p > \dim_{\mathrm{ARC}}(K, d)$, a combination of (b), [Kig23, Lemmas 3.13, 3.16, 3.19 and Theorem 3.21] and Theorem 8.16 implies that $(\widehat{\mathcal{E}}_p, \mathcal{W}^p)$ is a regular p-resistance form on K. Then the estimate (8.17) is exactly the same as [Kig23, (3.21) in Lemma 3.34], so we complete the proof. □

Remark 8.20. The construction of $\mathcal{E}_p^{\Gamma}$ in [MS23+, Theorem 6.22] is very similar to that of $\widehat{\mathcal{E}}_p$ in the proof above although the setting and assumption on a 'partition' in [MS23+] is slightly different from ours. Thanks to Proposition 8.7, the operators M_n and J_n defined in [MS23+, (6.8) and (6.9)] correspond to P_n and Q_n respectively. In particular, (8.19) and (8.20) for M_n and J_n are also true. Hence we can easily see that the p-energy form $(\mathcal{E}_p^{\Gamma}, \mathcal{F}_p)$ in [MS23+, Theorem 6.22] also satisfies $(\mathrm{GC})_p$.

Before concluding this subsection, we deal with the capacity upper bound and a Poincaré-type inequality under the additional assumption on the Ahlfors regularity of m. In addition to the density of $\mathcal{W}^p$ in $C(K)$, we can obtain the following capacity upper bound under the p-conductive homogeneity of K if $p > \dim_{\mathrm{ARC}}(K, d)$ and m is Ahlfors regular.

Proposition 8.21 (Capacity upper bound)**.** *Let $p \in (1, \infty)$ and $\lambda \in (1, \infty)$. Assume that Assumption 8.6 holds, that K is p-conductively homogeneous, that $p > \dim_{\mathrm{ARC}}(K, d)$ and that m is Ahlfors regular. Then there exists $C \in (0, \infty)$ such that for any $(x, r) \in K \times (0, 1]$,*

$$\inf\bigl\{\mathcal{N}_p(u)^p \bigm| u \in \mathcal{W}^p, u|_{B_d(x,r)} = 1, \mathrm{supp}_K[u] \subseteq B_d(x, \lambda r)\bigr\} \le C\frac{m(B_d(x, r))}{r^{d_{\mathrm{w},p}}}. \tag{8.22}$$

Proof. Let $r_* \in (0, 1)$ and $M_* \in \mathbb{N}$ be the constants in Assumption 8.6. For $r \in (0, 1]$, choose $n \in \mathbb{N}$ as the minimal number so that $c_2(M_* + 1)r_*^n < (\lambda - 1)r$, where c_2 is the constant in (8.3). Let $x \in K$ and set $T_n(x, r) := T_n[B_d(x, r)]$ for simplicity. Then, by the metric doubling property of (K, d), there exists $N \in \mathbb{N}$ which is independent of x and r such that $\#T_n(x, r) \le N$. By [Kig23, Lemma 3.18] and its proof, for any $w \in T_n(x, r)$ there exists $h_{M_*,w} \in \mathcal{W}^p$ such that $h_{M_*,w}|_{K_w} = 1$, $\mathrm{supp}_K[h_{M_*,w}] \subseteq U_{M_*}(w)$ and $\mathcal{N}_p(h_{M_*,w})^p \lesssim \sigma_p^n$. Now we define $\psi_{x,r} := \sum_{w \in T_n(x,r)} h_{M_*,w} \in \mathcal{W}^p$. Then $\psi_{x,r}|_{B_d(x,s)} \ge 1$, $\mathrm{supp}_K[\psi_{x,r}] \subseteq B_d(x, \lambda r)$ and

$$\mathcal{N}_p(\psi_{x,r})^p \le N^{p-1} \max_{w \in T_n(x,r)} \mathcal{N}_p(h_{M_*,w})^p \lesssim \sigma_p^n = r_*^{n(d_{\mathrm{f}} - d_{\mathrm{w},p})} \lesssim r^{d_{\mathrm{f}} - d_{\mathrm{w},p}}.$$

Since m is Ahlfors regular and $\mathcal{N}_p(\psi_{x,r} \wedge 1) \lesssim \mathcal{N}_p(\psi_{x,r})$ by [Kig23, Theorem 3,21], we obtain (8.22). □

We can describe Poincaré-type inequalities in terms of discrete p-energies as follows.

Lemma 8.22. *Let $p \in (1, \infty)$. Assume that Assumption 8.6 holds, that K is p-conductively homogeneous, and that m is Ahlfors regular. Then there exists a constant $C \in (0, \infty)$ such that for any $f \in L^p(K, m)$ and any $w \in T$,*

$$\int_{K_w} \left| f(x) - \fint_{K_w} f\,dm \right|^p m(dx) \le C r_*^{|w| d_{\mathrm{w},p}} \liminf_{n \to \infty} \widetilde{\mathcal{E}}_{p,S^n(w)}^{n+|w|}(f). \tag{8.23}$$

Proof. Set $k := |w|$. Recall that $\lim_{n\to\infty}\|Q_n f - f\|_{L^p(K,m)} = 0$ as shown in the proof of Theorem 8.19-(b). Hence, for any $n \in \mathbb{N}$, we see that

$$\begin{aligned}
&\frac{1}{m(K_w)}\sum_{v\in S^n(w)}|P_{n+k}f(v) - P_k f(w)|^p\, m(K_v) \\
&= \frac{1}{m(K_w)}\sum_{v\in S^n(w)}\int_{K_v}|Q_{n+k}f(x) - P_k f(w)|^p\, m(dx) \\
&= ⨍_{K_w}|Q_{n+k}f(x) - P_k f(w)|^p\, m(dx) \xrightarrow[n\to\infty]{} ⨍_{K_w}|f(x) - P_k f(w)|^p\, m(dx), \qquad (8.24)
\end{aligned}$$

where we used Proposition 8.7 in the second equalty. By [Kig23, (5.11) in Theorem 5.11] and (8.10), there exists $C \in (0,\infty)$ which is independent of f and n such that

$$\frac{1}{m(K_w)}\sum_{v\in S^n(w)}|P_{n+k}f(v) - P_k f(w)|^p\, m(K_v) \le C r_*^{k(d_{\mathrm{w},p}-d_{\mathrm{f}})}\widetilde{\mathcal{E}}_{p,S^n(w)}^{n+k}(f). \qquad (8.25)$$

We obtain (8.23) by combining (8.24), (8.25), (8.5) and the Ahlfors regularity of m. $\square$

Proposition 8.23. *Let $p \in (1,\infty)$. Assume that Assumption 8.6 holds, that K is p-conductively homogeneous, and that m is Ahlfors regular. Then there exist $C, \alpha \in (0,\infty)$ such that for any $(x,r) \in K \times (0,1]$ and any $f \in L^p(K,m)$,*

$$\int_{B_d(x,r)}\left|f - ⨍_{B_d(x,r)} f\, dm\right|^p dm \le C r^{d_{\mathrm{w},p}} \liminf_{k\to\infty}\widetilde{\mathcal{E}}_{p,T_k[B_d(x,\alpha r)]}^{k}(f). \qquad (8.26)$$

Proof. Throughout this proof, $M_* \in \mathbb{N}$ and $r_* \in (0,1)$ are the same constants as in Assumption 8.6. Let $(x,r) \in K \times (0,1]$. We first consider the case $r \in (c_3 r_*, 1]$, where c_3 is the constant in (8.4). By applying Lemma 8.22 for $w = \phi$,

$$\begin{aligned}
\int_{B_d(x,r)}\left|f - ⨍_{B_d(x,r)} f\, dm\right|^p dm &\overset{(7.14)}{\le} 2^p\int_{B_d(x,r)}\left|f - ⨍_{K} f\, dm\right|^p dm \\
&\le \int_{K}\left|f - ⨍_{K} f\, dm\right|^p dm \le C\liminf_{n\to\infty}\widetilde{\mathcal{E}}_p^n(f),
\end{aligned}$$

where $C \in (0,\infty)$ is the constant in (8.23). Since $\mathrm{diam}(K,d) = 1$, this shows (8.26) for any $A \ge c_3^{-1}r_*^{-1}$. Hence it suffices to consider the remaining case, i.e., $r \in (0, c_3 r_*]$. Let $n \in \mathbb{N}$ satisfy $c_3 r_*^n \ge r > c_3 r_*^{n+1}$,. Set $\Gamma_{M_*}(x;n) := \{v \in T \mid v \in \Gamma_{M_*}(w) \text{ for some } w \in T_n \text{ with } x \in K_w\}$. Then we see that

$$\begin{aligned}
&\int_{U_{M_*}(x;n)}\left|f(y) - ⨍_{U_{M_*}(x;n)} f\, dm\right|^p m(dy) \\
&\le 2^{p-1}\sum_{w\in\Gamma_{M_*}(x;n)}\left(\int_{K_w}|f(y) - P_n f(w)|^p\, m(dy) + m(K_w)\left|P_n f(w) - ⨍_{U_{M_*}(x;n)} f\, dm\right|^p\right)
\end{aligned}$$

$$\lesssim \sum_{w\in\Gamma_{M_*}(x;n)} \left(r^{d_{\mathrm{w},p}} \liminf_{k\to\infty} \widetilde{\mathcal{E}}^{n+k}_{p,S^k(w)}(f) + r^{d_{\mathrm{f}}} \left| P_n f(w) - ⨍_{U_{M_*}(x;n)} f\,dm \right|^p \right). \tag{8.27}$$

Note that, by Proposition 8.7,

$$P_n f(w) - ⨍_{U_{M_*}(x;n)} f\,dm = \frac{1}{m(U_{M_*}(x;n))} \sum_{v\in\Gamma_{M_*}(x;n)} (P_n f(w) - P_n f(v)) m(K_v).$$

For any $w \in \Gamma_{M_*}(x;n)$, by choosing $w' \in \Gamma_{M_*}(x;n) \setminus \{w\}$ so that $P_n f(w) - P_n f(w') = \max_{v\in\Gamma_{M_*}(x;n)} |P_n f(w) - P_n f(v)|$, we have

$$\left| P_n f(w) - ⨍_{U_{M_*}(x;n)} f\,dm \right| \le |P_n f(w) - P_n f(w')| .$$

Hence, by Hölder's inequality, (8.10) and [Kig23, (2.17)],

$$\begin{aligned}\left| P_n f(w) - ⨍_{U_{M_*}(x;n)} f\,dm \right|^p &\le (2M_* + 1)^{p-1} \mathcal{E}^n_{p,\Gamma_{M_*}(x;n)}(f) \\ &\lesssim r^{d_{\mathrm{w},p} - d_{\mathrm{f}}} \liminf_{k\to\infty} \widetilde{\mathcal{E}}^{n+k}_{p,S^k(\Gamma_{M_*}(x;n))}(f). \end{aligned} \tag{8.28}$$

Note that $\#\Gamma_{M_*}(x;n) \le L_*^{M_*+2}$ by Assumption 8.6-(1) and that $S^k(\Gamma_{M_*}(x;n)) \subseteq T_{n+k}[B_d(x, c_4 r_*^n)] \subseteq T_{n+k}[B_d(x, c_3^{-1} r_*^{-1} c_4 r)]$ by Assumption 8.6-(2), where c_4 is the same as in (8.4). Now we set $A := (1 \vee c_4) c_3^{-1} r_*^{-1}$. Then, by (8.27) and (8.28),

$$\begin{aligned}&\int_{U_{M_*}(x;n)} \left| f(y) - ⨍_{U_{M_*}(x;n)} f\,dm \right|^p m(dy) \\ &\overset{(8.28)}{\lesssim} r^{d_{\mathrm{w},p}} \liminf_{k\to\infty} \sum_{w\in\Gamma_{M_*}(x;n)} \widetilde{\mathcal{E}}^{n+k}_{p,S^k(\Gamma_{M_*}(x;n))}(f) \le L_*^{M_*+2} r^{d_{\mathrm{w},p}} \liminf_{k\to\infty} \widetilde{\mathcal{E}}^{k}_{p,T_k[B_d(x,Ar)]}(f).\end{aligned}$$

Since

$$\begin{aligned}\int_{B_d(z,s)} \left| f(y) - ⨍_{B_d(x,r)} f\,dm \right|^p m(dy) &\overset{(7.14)}{\le} 2^p \int_{B_d(z,s)} \left| f(y) - ⨍_{U_{M_*}(x;n)} f\,dm \right|^p m(dy) \\ &\overset{(8.4)}{\le} 2^p \int_{M_*(x;n)} \left| f(y) - ⨍_{U_{M_*}(x;n)} f\,dm \right|^p m(dy),\end{aligned}$$

we obtain (8.26). □

8.2 Construction of self-similar p-energy forms on p-conductively homogeneous self-similar structures

In this subsection, we construct a self-similar p-resistance form on self-similar structures under suitable assumptions. Our main result in this subsection, Theorem 8.29, implies that self-similar p-energy forms constructed in [Kig23, Theorem 4.6] satisfy $(\mathrm{GC})_p$.

We start with some preparations before constructing self-similar p-resistance forms. In the following definition, we introduce a good partition parametrized by a rooted tree.

Definition 8.24 ([Kig23, Definition 4.2])**.** Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a self-similar structure, let $r \in (0,1)$ and let $(j_s)_{s\in S} \in \mathbb{N}^S$. Define

$$j(w) \coloneqq \sum_{i=1}^{n} j_{w_i} \quad \text{and} \quad g(w) \coloneqq r^{j(w)} \quad \text{for } w = w_1 \dots w_n \in W_n.$$

Define $\widetilde{\pi}(w_1 \cdots w_n) \coloneqq w_1 \cdots w_{n-1}$ for $w = w_1 \dots w_n \in W_n$ and

$$\Lambda^g_{r^k} \coloneqq \{w = w_1 \cdots w_n \in W_* \mid g(\widetilde{\pi}(w)) > r^k \geq g(w)\}.$$

Set $T^{(r)}_k \coloneqq \{(k, w) \mid w \in \Lambda^g_{r^k}\}$, $T^{(r)} \coloneqq \bigcup_{k\in\mathbb{N}\cup\{0\}} T^{(r)}_k$ and define $\iota\colon T^{(r)} \to W_*$ as $\iota(k, w) = w$. Moreover, define $E_{T^{(r)}} \subseteq T^{(r)} \times T^{(r)}$ by

$$E_{T^{(r)}} \coloneqq \left\{((k, v), (k + 1, w)) \in T^{(r)}_k \times T^{(r)}_{k+1} \;\middle|\; k \in \mathbb{N} \cup \{0\}, v = w \text{ or } v = \widetilde{\pi}(w)\right\},$$

so that $(T^{(r)}, E_{T^{(r)}})$ is a rooted tree (see [Kig23, Proposition 4.3]).

In the rest of this subsection, we presume the following assumption on the geometry of our self-similar structure.

Assumption 8.25. Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a self-similar structure such that $\#S \geq 2$ and K is connected. Set $K_w \coloneqq K_{\iota(w)}$ for $w \in T^{(r_*)}_*$ for simplicity. There exist $r_* \in (0, 1)$ and a metric d giving the original topology of K with $\mathrm{diam}(K, d) = 1$ such that $(K, d, \{K_w\}_{w\in T^{(r_*)}}, m)$ satisfies Assumption 8.6, where m is the self-similar measure on K with weight $(r_*^{j_s d_{\mathrm{f}}})_{s\in S}$ and d_{f} is the unique number satisfying $\sum_{s\in S} r_*^{j_s d_{\mathrm{f}}} = 1$.

Under Assumption 8.25, we have the d_{f}-Ahlfors regularity of m as follows.

Proposition 8.26 ([Kig23, Proposition 4.5])**.** *The value d_{f} coincides with the Hausdorff dimension of (K, d) and m is d_{f}-Ahlfors regular with respect to d.*

To obtain a self-similar p-energy form on $\mathcal{L}$, we first discuss the self-similarity for $\mathcal{W}^p$ (recall (5.5)). The following lemma can be shown in exactly the same way as [Kig23, Theorem 4.6-(1)] although the condition $p > \dim_{\mathrm{ARC}}(K, d)$ is assumed in [Kig23, Theorem 4.6].

Lemma 8.27. *For any $u \in L^p(K, m)$, any $k \in \mathbb{N} \cup \{0\}$ and any $n \in \mathbb{N} \cup \{0\}$ with $n \geq \max_{w\in W_k} j(w)$,*

$$\sum_{w\in W_k} \mathcal{E}^{n-j(w)}_p(P_{n-j(w)}(u \circ F_w)) \leq \mathcal{E}^n_p(P_n u). \tag{8.29}$$

In particular, if in addition K is p-conductively homogeneous (with respect to $\{K_w\}_{w\in T^{(r_*)}}$), then $u \circ F_w \in \mathcal{W}^p$ for any $u \in \mathcal{W}^p$ and any $w \in W_*$, and hence

$$\mathcal{W}^p \cap C(K) \subseteq \{u \in C(K) \mid u \circ F_i \in \mathcal{W}^p \text{ for any } i \in S\}. \tag{8.30}$$

Similar to the case $p = 2$ (see, e.g., [Kig00, KZ92]), we will obtain a self-similar p-energy form on $(\mathcal{L}, m)$ with weight $\boldsymbol{\sigma}_p := (\sigma_p^{j_s})_{s\in S}$ as a fixed point obtained by applying Theorem 5.21. To this end, we need the converse inclusion of (8.30) and uniform estimates on $\mathcal{S}_{\boldsymbol{\sigma_p},n}(E)$ for any/some $E \in \mathcal{U}_p(\mathcal{W}^p \cap C(K))$; recall the definition of $\mathcal{S}_{\boldsymbol{\sigma}_p,n}$ in Definition 5.20. These conditions are true if K is p-conductively homogeneous and $p > \dim_{\mathrm{ARC}}(K, d)$ as described in the following proposition. (This result is essentially proved in [Kig23, Proof of Theorem 4.6].)

Proposition 8.28. *Let $p \in (1, \infty)$ and assume that K is p-conductively homogeneous (with respect to $\{K_w\}_{w\in T^{(r_*)}}$). If $p > \dim_{\mathrm{ARC}}(K, d)$, then*

$$\mathcal{W}^p = \{u \in C(K) \mid u \circ F_i \in \mathcal{W}^p \text{ for any } i \in S\}, \tag{8.31}$$

and there exists $C \in [1, \infty)$ such that for any $E \in \mathcal{U}_p$, any $u \in \mathcal{W}^p$ and any $n \in \mathbb{N}$,

$$C^{-1}\mathcal{N}_p(u)^p \leq \mathcal{S}_{\boldsymbol{\sigma_p},n}(E)(u) \leq C\mathcal{N}_p(u)^p. \tag{8.32}$$

Proof. The uniform estimate (8.32) follows from [Kig23, (4.6) and (4.8)]. (In the proof of [Kig23], the assumption $p > \dim_{\mathrm{ARC}}(K, d)$ is used to obtain [Kig23, (4.8)].) In the rest of the proof, we prove

$$\mathcal{W}^p \supseteq \{u \in C(K) \mid u \circ F_i \in \mathcal{W}^p \text{ for any } i \in S\} =: \mathcal{W}^p_S.$$

(The converse inclusion is proved in Lemma 8.27.) We note that the following estimate in [Kig23, lines 8-9 in p. 61] is true for every $u \in \mathcal{W}^p_S$: there exists a constant $C' \in (0, \infty)$ such that

$$\widetilde{\mathcal{E}}_p^n(u) \leq C' \sum_{w\in W_n} \sigma_p^{j(w)} \mathcal{N}_p(u \circ F_w)^p = C'\mathcal{S}_{\boldsymbol{\sigma}_p,n}(\mathcal{N}_p^p)(u) \quad \text{for any } n \in \mathbb{N},\ u \in \mathcal{W}^p_S. \tag{8.33}$$

(We need $p > \dim_{\mathrm{ARC}}(K, d)$ to obtain (8.33) by following the argument in [Kig23, p. 61].) Taking the supremum over $n \in \mathbb{N}$ in the left-hand side of (8.33), we have $\mathcal{W}^p_S \subseteq \mathcal{W}^p$. □

Now we can obtain the desired self-similar p-energy form. The following theorem is a generalization of [Kig23, Theorem 4.6] taking into account the case $p \leq \dim_{\mathrm{ARC}}(K, d)$.

Theorem 8.29. *Let $p \in (1, \infty)$. Assume that Assumption 8.25 holds, that K is p-conductively homogeneous (with respect to $\{K_w\}_{w\in T^{(r_*)}}$) and that the following* pre-self-similarity conditions *hold:*

$$\mathcal{W}^p \cap C(K) = \{u \in C(K) \mid u \circ F_i \in \mathcal{W}^p \text{ for any } i \in S\}. \tag{8.34}$$

There exists $C \in [1, \infty)$ such that (8.32) holds for any $u \in \mathcal{W}^p \cap C(K)$, $n \in \mathbb{N}$. (8.35)

Let σ_p be the constant in (8.11), set $\boldsymbol{\sigma}_p := (\sigma_p^{j_s})_{s\in S}$, let $(\widehat{\mathcal{E}}_p, \mathcal{W}^p)$ be any p-energy form on (K, m) given in Theorem 8.19 and set $\mathcal{F}_p := \overline{\mathcal{W}^p \cap C(K)}^{\mathcal{W}^p}$. Then there exists $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $n_k < n_{k+1}$ for any $k \in \mathbb{N}$ such that the following limit exists in $[0, \infty)$ for any $u \in \mathcal{F}_p$:

$$\mathcal{E}_p(u) := \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \mathcal{S}_{\boldsymbol{\sigma}_p,j}(\widehat{\mathcal{E}}_p)(u). \tag{8.36}$$

Moreover, the following properties hold:

(a) $(\mathcal{E}_p, \mathcal{F}_p)$ *is a self-similar p-energy form on $(\mathcal{L}, m)$ with weight $\boldsymbol{\sigma}_p$, and there exist $\alpha_0, \alpha_1 \in (0, \infty)$ such that $\alpha_0 \mathcal{N}_p(u)^p \leq \mathcal{E}_p(u) \leq \alpha_1 \mathcal{N}_p(u)^p$ for any $u \in \mathcal{F}_p$.*

(b) (Generalized p-contraction property) $(\mathcal{E}_p, \mathcal{F}_p)$ *satisfies* (GC)$_p$.

(c) (Strong locality) $(\mathcal{E}_p, \mathcal{F}_p)$ *satisfies the strongly local property* (SL1).

(d) *If in addition $p > \dim_{\mathrm{ARC}}(K, d)$, then $\mathcal{F}_p = \mathcal{W}^p$ and $(\mathcal{E}_p, \mathcal{F}_p)$ is a regular self-similar p-resistance form on $\mathcal{L}$ with weight $\boldsymbol{\sigma}_p$ and there exist $\alpha_0, \alpha_1 \in (0, \infty)$ such that*

$$\alpha_0\, d(x, y)^{\tau_p} \leq R_{\mathcal{E}_p}(x, y) \leq \alpha_1\, d(x, y)^{\tau_p} \quad \text{for any } x, y \in K. \tag{8.37}$$

Remark 8.30. (1) In the case $p > \dim_{\mathrm{ARC}}(K, d)$, the pre-self-similarity conditions, (8.34) and (8.35), can be dropped by virtue of Proposition 8.28.

(2) In [CGQ22], self-similar p-energy forms on p.-c.f. self-similar structures are constructed, which are p-resistance forms under a certain condition as shown in Subsection 8.3. Note that *any* $p \in (1, \infty)$ is allowed in the framework of [CGQ22] unlike that of [Kig23] (see (d) above). It is extremely hard to determine the value $\dim_{\mathrm{ARC}}(K, d)$ in general; however, $\dim_{\mathrm{ARC}}(K, d)$ for a p.-c.f. self-similar set K is typically 1. (See [CP14, Theorem 1.2] for a sufficient condition for $\dim_{\mathrm{ARC}}(K, d) = 1$.) In Appendix B.2, by using some results in [CGQ22], we prove that the Ahlfors regular conformal dimension of any *affine nested fractal* equipped with the p-resistance metric is 1.

Proof. The existence of the limit in (8.36) and its properties (a), (b) and (c) are immediate from (8.34), (8.32), Lemma 5.15, Theorem 5.21, Propositions 5.22-(a) and 5.23. Let us verify (d). Recall that $\mathcal{W}^p \subseteq C(K)$ by $p > \dim_{\mathrm{ARC}}(K, d)$ (see Theorem 8.16), whence $\mathcal{F}_p = \mathcal{W}^p$. A similar argument as in the proof of Theorem 8.19-(d) shows that $(\mathcal{E}_p, \mathcal{W}^p)$ is a regular p-resistance form on K satisfying (8.37). This completes the proof. □

Similar to Theorem 7.9, we can obtain the monotonicity of $\sigma_p^{1/(p-1)}$ in $p > \dim_{\mathrm{ARC}}(K, d)$. Note that the following result is *not* restricted to p.-c.f. self-similar structures.

Theorem 8.31. *Assume that Assumption 8.25 holds. Let $p, q \in (\dim_{\mathrm{ARC}}(K, d), \infty)$ with $p \leq q$. In addition, assume that K is s-conductively homogeneous (with respect to $\{K_w\}_{w \in T^{(r_*)}}$) for each $s \in \{p, q\}$. Then*

$$\sigma_p^{1/(p-1)} \leq \sigma_q^{1/(q-1)}. \tag{8.38}$$

Proof. The proof is very similar to that of Theorem 7.9. By Proposition 8.28, (8.34) and (8.32) with $s \in \{p, q\}$ in place of p hold. Let $(\mathcal{E}_s, \mathcal{W}^s)$ be a self-similar s-resistance form on $\mathcal{L}$ given in Theorem 8.29 for each $s \in \{p, q\}$. Fix two distinct points $x_0, y_0 \in K$, set $B \coloneqq \{x_0, y_0\}$ and define $h_p \coloneqq h_B^{\mathcal{E}_p}\bigl[\mathbb{1}_{x_0}^B\bigr] \in \mathcal{W}^p$. Then $0 \leq h_p \leq 1$ by the weak comparison principle (Proposition 6.26) and we can find $w \in W_*$ satisfying $K_w \cap B = \emptyset$ and $h_{p,w} \coloneqq h_p \circ F_w \notin \mathbb{R}\mathbb{1}_K$. Similar to (7.13), by using (6.31) and (7.1), for any $\{u, v\} \in E_n^*$, we can show that

$$|P_n h_{q,w}(u) - P_n h_{q,w}(v)|^{q-p} \leq C r_*^{n(d_{\mathrm{w},p} - d_{\mathrm{f}})\frac{q-p}{p-1}},$$

where $C \in (0,\infty)$ is independent of n. Hence we have

$$\mathcal{E}_q^n(h_{p,w}) = \sum_{\{u,v\}\in E_n^*} |P_n h_{q,w}(u) - P_n h_{q,w}(v)|^q \le C r_*^{n(d_{\mathrm{w},p}-d_{\mathrm{f}})\frac{q-p}{p-1}} \mathcal{E}_p^n(h_{p,w}),$$

which implies that

$$\left(\sigma_q^{-1}\sigma_p^{(q-1)/(p-1)}\right)^n \widetilde{\mathcal{E}}_q^n(h_{p,w}) \le C\widetilde{\mathcal{E}}_p^n(h_{p,w}) \le C\mathcal{N}_p(h_{p,w})^p. \tag{8.39}$$

By (8.13), there exists $C_q \in (0,\infty)$ such that $\mathcal{N}_q(f)^q \le C_q \liminf_{n\to\infty} \widetilde{\mathcal{E}}_q^n(f)$ for any $f \in L^q(K,m)$. This together with (8.39) implies that

$$\mathcal{N}_q(h_{p,w})^q \limsup_{n\to\infty}\left(\sigma_q^{-1}\sigma_p^{(q-1)/(p-1)}\right)^n \le C'\mathcal{N}_p(h_{p,w})^p < \infty.$$

Since $\mathcal{N}_q(h_{p,w}) > 0$, we obtain $\sigma_q^{-1}\sigma_p^{(q-1)/(p-1)} \le 1$, which yields (8.38). □

We conclude this subsection by applying Theorem 6.37 (elliptic Harnack inequality) in the case $p > \dim_{\mathrm{ARC}}(K,d)$ of Theorem 8.29. We immediately obtain the following corollary by combining Propositions 7.12, 8.21, 8.26 and (8.37).

Corollary 8.32 (Elliptic Harnack inequality for self-similar p-resistance form)**.** *Let $p \in (1,\infty)$. Assume that Assumption 8.25 holds, that K is p-conductively homogeneous (with respect to $\{K_w\}_{w\in T^{(r_*)}}$) and that $p > \dim_{\mathrm{ARC}}(K,d)$. Then $(\mathcal{E}_p, \mathcal{W}^p)$ and $\{\Gamma_{\mathcal{E}_p}\langle u\rangle\}_{u\in\mathcal{W}^p}$ given in Theorem 8.29 and in (5.11) respectively satisfy the assumptions in Theorem 6.37 with m, $\frac{d_{\mathrm{f}}(p-1)}{\tau_p}$, $\frac{d_{\mathrm{w},p}(p-1)}{\tau_p}$ in place of μ, Q, β.*

8.3 Construction of self-similar p-resistance forms on post-critically finite self-similar structures

In this subsection, under the condition (**R**) of [CGQ22, p. 18], we see that the construction of p-energy forms on p.-c.f. self-similar structures constructed due to [CGQ22] yields p-resistance forms. The framework in [CGQ22] is focused only on p.-c.f. self-similar structures, but restrictions on weights of self-similar p-resistance forms are flexible so that non-arithmetic weights are allowed unlike the framework in Subsection 8.2. See Section B.1 for details.

In the following definitions, we recall some classes of p-energy forms on finite sets considered in [CGQ22].

Definition 8.33 ([CGQ22, Definition 2.1])**.** Let A be a finite set with $\#A \ge 2$. Let $E\colon \mathbb{R}^A \to [0,\infty)$ and consider the following conditions.

(i) $E(tf + (1-t)g) \le tE(f) + (1-t)E(g)$ for any $f, g \in \mathbb{R}^A$ and any $t \in [0,1]$.

(ii) $E(tf) = |t|^p E(f)$ for any $f \in \mathbb{R}^A$ and any $t \in \mathbb{R}$.

(iii) $E(f + t\mathbb{1}_A) = E(f)$ for any $f \in \mathbb{R}^A$ and any $t \in \mathbb{R}$.

(iv) $E(f^{+} \wedge 1) \leq E(f)$ for any $f \in \mathbb{R}^{A}$.
(v) $\{f \in \mathbb{R}^{A} \mid E(f) = 0\} = \mathbb{R}\mathbb{1}_{A}$.

We define $\mathcal{M}_p(A)$ and $\widetilde{\mathcal{M}}_p(A)$ by

$$\mathcal{M}_p(A) := \{E\colon \mathbb{R}^{A} \to [0,\infty) \mid E \text{ satisfies (i)-(v)}\}, \tag{8.40}$$
$$\widetilde{\mathcal{M}}_p(A) := \{E\colon \mathbb{R}^{A} \to [0,\infty) \mid E \text{ satisfies (i)-(iv)}\}. \tag{8.41}$$

Definition 8.34 ([CGQ22, Definition 2.8]). Let A be a finite set with $\#A \geq 2$. For $E_1, E_2 \in \widetilde{\mathcal{M}}_p(A)$, define a metric $d_{\widetilde{\mathcal{M}}_p(A)}$ on $\widetilde{\mathcal{M}}_p(A)$ by

$$d_{\widetilde{\mathcal{M}}_p(A)}(E_1, E_2) := \sup\Bigl\{|E_1(u) - E_2(u)| \Bigm| u \in \mathbb{R}^{A}, \operatorname*{osc}_{A}[u] = 1\Bigr\}. \tag{8.42}$$

For simplicity, we set $|E|_{\widetilde{\mathcal{M}}_p(A)} := d_{\widetilde{\mathcal{M}}_p(A)}(E, 0)$ for $E \in \widetilde{\mathcal{M}}_p(A)$.

(1) We define $\mathcal{S}_p(A) \subseteq \mathcal{M}_p(A)$ by

$$\mathcal{S}_p(A) := \left\{E \in \mathcal{M}_p(A) \;\middle|\; \begin{array}{l}\text{there exists } (c_{xy})_{x,y\in A} \subseteq [0,\infty) \text{ such that}\\ E(f) = \sum_{x,y\in A} |f(x) - f(y)|^{p}\, c_{xy} \text{ for } f \in \mathbb{R}^{A}\end{array}\right\}. \tag{8.43}$$

Note that any functional in $\mathcal{S}_p(A)$ is a p-resistance form on A (see Example 6.3-(3)).

(2) We define $\mathcal{Q}'_p(A) \subseteq \mathcal{M}_p(A)$ by

$$\mathcal{Q}'_p(A) := \left\{E \in \mathcal{M}_p(A) \;\middle|\; \begin{array}{l}\text{there exist } B \supseteq A \text{ and } \widetilde{E} \in \mathcal{S}_p(B) \text{ such that}\\ \widetilde{E}\big|_{A} = E, \text{ where } \widetilde{E}\big|_{A} \text{ is the trace of } \widetilde{E} \text{ on } A\end{array}\right\}. \tag{8.44}$$

Let $\mathcal{Q}_p(A)$ be the closure of $\mathcal{Q}'_p(A)$ in $(\mathcal{M}_p(A), d_{\widetilde{\mathcal{M}}_p(A)})$, i.e.,

$$\mathcal{Q}_p(A) := \left\{E \in \mathcal{M}_p(A) \;\middle|\; \begin{array}{l}\text{there exists } \{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{Q}'_p(A) \text{ such}\\ \text{that } \lim_{n\to\infty} d_{\widetilde{\mathcal{M}}_p(A)}(E, E_n) = 0\end{array}\right\}. \tag{8.45}$$

Then we can show that any functional in $\mathcal{Q}_p(A)$ is a p-resistance form on A.

Proposition 8.35. *Let A be a finite set with $\#A \geq 2$ and let $E \in \mathcal{Q}_p(A)$. Then E is a p-resistance form on A.*

Proof. Thanks to Proposition 2.9-(a), it suffices to prove $(\mathrm{RF5})_p$, i.e., $(\mathrm{GC})_p$, for $E \in \mathcal{Q}_p(A)$. Let $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{Q}'_p(A)$ satisfy $\lim_{n\to\infty} d_{\widetilde{\mathcal{M}}_p(A)}(E, E_n) = 0$. Then it is easy to see that $\lim_{n\to\infty} E_n(u) = E(u)$ for any $u \in \mathbb{R}^{A}$ (see also [CGQ22, Lemma A.1]). Since E_n satisfies $(\mathrm{GC})_p$ for any $n \in \mathbb{N}$, we have $(\mathrm{GC})_p$ for E by Proposition 2.9-(b). □

Next we introduce renormalization operators playing central roles in the construction of p-energy forms on p.-c.f. self-similar structures. In the rest of this subsection, we always suppose that K is connected and that $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is a p.-c.f. self-similar structure with $\#S \geq 2$.

Definition 8.36 (Renormalization operator; [CGQ22, Definition 3.1])**.** Let $\boldsymbol{\rho}_p = (\rho_{p,i})_{i\in S} \in (0,\infty)^S$ and $k \in \mathbb{N}\cup\{0\}$. For a p-resistance form E on V_k, define p-resistance forms $\Lambda_{\boldsymbol{\rho}_p}(E)\colon \mathbb{R}^{V_{k+1}} \to [0,\infty)$ and $\mathcal{R}_{\boldsymbol{\rho}_p}(E)\colon \mathbb{R}^{V_k} \to [0,\infty)$[9] by

$$\Lambda_{\boldsymbol{\rho}_p}(E)(u) \coloneqq \sum_{i\in S}\rho_{p,i}E(u\circ F_i), \quad u\in\mathbb{R}^{V_{k+1}}, \quad \text{and} \quad \mathcal{R}_{\boldsymbol{\rho}_p}(E) \coloneqq \Lambda_{\boldsymbol{\rho}_p}(E)\big|_{V_k}. \tag{8.46}$$

(Recall Proposition 7.8 and Theorem 6.13.) Precisely, $\Lambda_{\boldsymbol{\rho}_p}, \mathcal{R}_{\boldsymbol{\rho}_p}$ depend on k, but we omit it for convenience. By [CGQ22, Lemma 3.2-(b)], we have $\Lambda^n_{\boldsymbol{\rho}_p}(E)\big|_{V_k} = \mathcal{R}^n_{\boldsymbol{\rho}_p}(E)$ for any $n\in\mathbb{N}\cup\{0\}$, i.e.,

$$\mathcal{R}^n_{\boldsymbol{\rho}_p}(E)(u) = \inf\left\{\sum_{w\in W_n}\rho_{p,w}E(v\circ F_w) \;\middle|\; v\in\mathbb{R}^{V_{n+k}}, v|_{V_k} = u\right\}, \quad u\in\mathbb{R}^{V_k}.$$

The following theorem ensures the existence of an eigenform with respect to $\mathcal{R}_{\boldsymbol{\rho}_p}$. This theorem can be shown by combining [CGQ22, Lemma 4.4, proof of Theorem 4.2] and Proposition 8.35, so we omit the proof.

Theorem 8.37 (Existence of an eigenform)**.** *Let $\boldsymbol{\rho}_p = (\rho_{p,i})_{i\in S} \in (0,\infty)^S$. Assume that there exist $c\in(0,\infty)$ and a p-resistance form E on V_0 such that*

$$\min_{x,y\in V_0; x\neq y} R_{\mathcal{R}^n_{\boldsymbol{\rho}_p}(E)}(x,y) \geq c \max_{x,y\in V_0; x\neq y} R_{\mathcal{R}^n_{\boldsymbol{\rho}_p}(E)}(x,y) \quad \textit{for any } n\in\mathbb{N}\cup\{0\}. \tag{\textbf{A}}$$

(a) *There exists a unique number $\lambda = \lambda(\boldsymbol{\rho}_p) \in (0,\infty)$ such that the following hold. For any $E' \in \mathcal{M}_p(V_0)$, there exists $C\in[1,\infty)$ such that*

$$C^{-1}\lambda^n E'(u) \leq \mathcal{R}^n_{\boldsymbol{\rho}_p}(E')(u) \leq C\lambda^n E'(u) \quad \textit{for any } n\in\mathbb{N}\cup\{0\} \textit{ and any } u\in\mathbb{R}^{V_0}. \tag{8.47}$$

(b) *Let $E_0 \in \mathcal{S}_p(V_0)$. For $n\in\mathbb{N}$, define $E_n \in \mathcal{Q}'_p(V_0)$ by*

$$E_n(u) \coloneqq \inf\left\{\frac{1}{n+1}\sum_{j=0}^{n}\lambda^{-j}\Lambda^j_{\boldsymbol{\rho}_p}(E_0)(v|_{V_j}) \;\middle|\; v\in\mathbb{R}^{V_n}, v|_{V_0} = u\right\}, \quad u\in\mathbb{R}^{V_0}, \tag{8.48}$$

where λ is the number given in (a). Then there exists a subsequence $\{E_{n_k}\}_{k\in\mathbb{N}}$ such that it converges in the topology induced by $d_{\widetilde{\mathcal{M}}_p}$. In particular, there exists $E_* \in \mathcal{Q}_p(V_0)$ such that

$$E_*(u) = \lim_{k\to\infty}\frac{1}{n_k+1}\sum_{j=0}^{n_k}\lambda^{-j}\Lambda^j_{\boldsymbol{\rho}_p}(E_0)(u), \quad u\in\mathbb{R}^{V_0}. \tag{8.49}$$

(c) *Let $E_0 \in \mathcal{S}_p(V_0)$, let $E_* \in \mathcal{Q}_p(V_0)$ be given by (8.49) and let λ be the number given in (a). Then $\{\lambda^{-l}\mathcal{R}^l_{\boldsymbol{\rho}_p}(E_*)(u)\}_{l\in\mathbb{N}\cup\{0\}}$ is non-decreasing for any $u\in\mathbb{R}^{V_0}$ and $\mathcal{R}_{\boldsymbol{\rho}_p}(\mathcal{E}^{(0)}_p) = \lambda\mathcal{E}^{(0)}_p$, where*

$$\mathcal{E}^{(0)}_p(u) \coloneqq \lim_{l\to\infty}\lambda^{-l}\mathcal{R}^l_{\boldsymbol{\rho}_p}(E_*)(u), \quad u\in\mathbb{R}^{V_0}. \tag{8.50}$$

[9] We use different symbols from [CGQ22].

Remark 8.38. If $\boldsymbol{\rho}_p$ satisfies (**A**) for some p-resistance form E on V_0, then for any p-resistance form $\widetilde{E}$ on V_0 there exists $\tilde{c} \in (0, \infty)$ such that (**A**) with $\widetilde{E}, \tilde{c}$ in place of E, c holds by [CGQ22, Lemma 4.4-(a)]. Hence (**A**) is a condition relying only on $\boldsymbol{\rho}_p$.

In the rest of this subsection, we fix $\boldsymbol{\rho}_p = (\rho_{p,i})_{i \in S} \in (0, \infty)^S$. Let us introduce two important conditions on $\boldsymbol{\rho}_p$ similarly to [CGQ22].

(**A**') There exist a p-resistance form $\mathcal{E}_p^{(0)}$ on V_0 such that $\mathcal{R}_{\boldsymbol{\rho}_p}(\mathcal{E}_p^{(0)}) = \mathcal{E}_p^{(0)}$.
(**R**) ((**A**') holds and) $\min_{i \in S} \rho_{p,i} > 1$.

Note that, by Theorem 8.37, (**A**') is equivalent to (**A**).

The following proposition is important to construct a self-similar p-resistance form as an "inductive limit" of discrete p-resistance forms as presented in [CGQ22, Proposition 5.3], which is an adaptation of the relevant pieces of the theory of resistance forms due to [Kig01, Sections 2.2, 2.3 and 3.3].

Proposition 8.39. *Assume that* (**A**') *holds. We define* $\mathcal{E}_p^{(n)} \coloneqq \Lambda_{\boldsymbol{\rho}_p}^n(\mathcal{E}_p^{(0)})$*, i.e.,*

$$\mathcal{E}_p^{(n)}(u) \coloneqq \sum_{w \in W_n} \rho_{p,w} \mathcal{E}_p^{(0)}(u \circ F_w), \quad u \in \mathbb{R}^{V_n}. \tag{8.51}$$

Then $\mathcal{E}_p^{(n)}$ *is a* p*-resistance form on* V_n *and* $\mathcal{E}_p^{(n+m)}|_{V_n} = \mathcal{E}_p^{(n)}$ *for any* $n, m \in \mathbb{N} \cup \{0\}$*, i.e.,* $\{(V_n, \mathcal{E}_p^{(n)})\}_{n \geq 0}$ *is a compatible sequence of* p*-resistance forms.*

Proof. We will show $\mathcal{E}_p^{(n+m)}|_{V_n} = \mathcal{E}_p^{(n)}$. (See [Kig01, Proposition 3.1.3] for the case $p = 2$.) It suffices to prove $\mathcal{E}_p^{(n+1)}|_{V_n} = \mathcal{E}_p^{(n)}$ for any $n \in \mathbb{N} \cup \{0\}$ by virtue of Proposition 6.15. Note that the case $n = 0$ is true by $\mathcal{R}_{\boldsymbol{\rho}_p}(\mathcal{E}_p^{(0)}) = \mathcal{E}_p^{(0)}$, and that

$$\mathcal{E}_p^{(n+1)}(u) = \sum_{i \in S} \rho_{p,i} \mathcal{E}_p^{(n)}(u \circ F_i), \quad \text{for any } n \in \mathbb{N} \cup \{0\} \text{ and } u \in \mathbb{R}^{V_{n+1}}. \tag{8.52}$$

Assume that $\mathcal{E}_p^{(m)}|_{V_{m-1}} = \mathcal{E}_p^{(m-1)}$ for some $m \in \mathbb{N}$. Then for any $u \in \mathbb{R}^{V_m}$,

$$\begin{aligned} \mathcal{E}_p^{(m)}(u) &\overset{(8.52)}{=} \sum_{i \in S} \rho_{p,i} \mathcal{E}_p^{(m-1)}(u \circ F_i) \\ &= \sum_{i \in S} \rho_{p,i} \min\left\{ \mathcal{E}_p^{(m)}(v \circ F_i) \;\middle|\; v \in \mathbb{R}^{K_i \cap V_{m+1}}, v|_{K_i \cap V_m} = u|_{K_i} \right\} \\ &\overset{(5.2)}{=} \min\left\{ \sum_{i \in S} \rho_{p,i} \mathcal{E}_p^{(m)}(v \circ F_i) \;\middle|\; v \in \mathbb{R}^{V_{m+1}}, v|_{V_m} = u \right\} \\ &\overset{(8.52)}{=} \min\left\{ \mathcal{E}_p^{(m+1)}(v) \;\middle|\; v \in \mathbb{R}^{V_{m+1}}, v|_{V_m} = u \right\} = \mathcal{E}_p^{(m+1)}|_{V_m}(u), \end{aligned}$$

which completes the proof. □

We can naturally construct a p-resistance form as an inductive limit on the countable set V_* as described in the following proposition.

Proposition 8.40. *Assume that* (**A'**) *holds. We define a linear subspace $\mathcal{F}_{p,*}$ of $\mathbb{R}^{V_*}$ and $\mathcal{E}_{p,*}\colon \mathcal{F}_{p,*} \to [0,\infty)$ by*

$$\mathcal{F}_{p,*} := \Bigl\{ u \in \mathbb{R}^{V_*} \Bigm| \lim_{n\to\infty} \mathcal{E}_p^{(n)}(u|_{V_n}) < \infty \Bigr\}, \quad \textit{and} \tag{8.53}$$

$$\mathcal{E}_{p,*}(u) := \lim_{n\to\infty} \mathcal{E}_p^{(n)}(u|_{V_n}), \quad u \in \mathcal{F}_{p,*}. \tag{8.54}$$

Then $(\mathcal{E}_{p,*}, \mathcal{F}_{p,*})$ is a p-resistance form on V_* satisfying $\mathcal{E}_{p,*}|_{V_n} = \mathcal{E}_{p,*}^{(n)}$ for any $n \in \mathbb{N}\cup\{0\}$. Moreover, the following self-similar properties hold:

$$\mathcal{F}_{p,*} = \bigl\{ u \in \mathbb{R}^{V_*} \bigm| u \circ F_i \in \mathcal{F}_{p,*} \textit{ for any } i \in S \bigr\}, \tag{8.55}$$

$$\mathcal{E}_{p,*}(u) = \sum_{i\in S} \rho_{p,i}\mathcal{E}_{p,*}(u \circ F_i) \quad \textit{for any } u \in \mathcal{F}_{p,*}. \tag{8.56}$$

If in addition (**R**) *holds, then for any $u \in \mathcal{F}_{p,*}$ there exists a unique $\widehat{u} \in C(K)$ such that $\widehat{u}|_{V_*} = u$, and $\{\widehat{u} \mid u \in \mathcal{F}_{p,*}\}$ is dense in $C(K)$.*

Proof. It is immediate from Theorem 6.21 that $(\mathcal{E}_{p,*}, \mathcal{F}_{p,*})$ is a p-resistance form on V_* with $\mathcal{E}_{p,*}|_{V_n} = \mathcal{E}_{p,*}^{(n)}$. By the definition in (8.51), it is easy to see that for any $n, k \in \mathbb{N}\cup\{0\}$ and any $u \in \mathbb{R}^{V_*}$,

$$\mathcal{E}_p^{(n+k)}(u|_{V_{n+k}}) = \sum_{w\in W_k} \rho_{p,w}\mathcal{E}_p^{(n)}(u \circ F_w|_{V_n}).$$

This immediately implies (8.55) and (8.56). The existence of unique continuous extensions of functions in $\mathcal{F}_{p,*}$ under (**R**) is proved in [CGQ22, Theorem 5.1-(b)]. A standard argument using the Stone–Weierstrass theorem shows that $\mathscr{C} := \{\widehat{u} \mid u \in \mathcal{F}_{p,*}\}$ is dense in $C(K)$. Indeed, $\mathscr{C}$ is an algebra since $\mathcal{F}_{p,*}$ is also an algebra by Proposition 2.2-(d). For any $x, y \in K$ with $x \neq y$, choose $n \in \mathbb{N}$ and $v, w \in W_n$ so that $x \in K_v, y \in K_w$ and $K_v \cap K_w = \emptyset$. (Such n, v, w exist by (5.3).) Then, by setting $v := h_{V_n}^{\mathcal{E}_{p,*}}[\mathbb{1}_{F_v(V_0)}]$, we see that $\varphi_{xy} := \widehat{v} \in \mathscr{C}$ satisfies $\varphi_{xy}(x) = 1$ and $\varphi_{xy}(y) = 0$, so we can use the Stone–Weierstrass theorem to conclude that $\mathscr{C}$ is dense in $C(K)$. $\square$

To extend $(\mathcal{E}_{p,*}, \mathcal{F}_{p,*})$ to a p-energy form defined on K, we need to specify how to regard functions in $\mathcal{F}_{p,*}$ as functions defined on K, which is indeed a delicate problem and discussed in [CGQ22, Theorems 5.1 and 5.2]. In this paper, we are mainly interested in the case $\mathcal{F}_{p,*}$ can be embedded into $C(K)$. In other words, we always assume that (**R**) holds. (See [CGQ22, Theorem 5.2] and [KS.b, Appendix] for details on a situation when we can identify a function $u \in \mathbb{R}^{V_*}$ satisfying $\lim_{n\to\infty} \mathcal{E}_p^{(n)}(u|_{V_n}) < \infty$ with a function on K without (**R**).) To state a construction of self-similar p-resistance forms under (**R**), we need the following lemma.

Lemma 8.41. *Assume that* (**A'**) *and* (**R**) *hold. Let $(\mathcal{E}_{p,*}, \mathcal{F}_{p,*})$ is the p-resistance form on V_* given in Proposition 8.40. Then $\mathrm{id}_{V_*}\colon (V_*, R_{\mathcal{E}_{p,*}}^{1/p}) \to K$ is uniquely extended to the completion, which gives a homeomorphism.*

Proof. The proof is very similar to arguments in [Kig01, Proposition 3.3.2, Lemma 3.3.5 and Theorem 3.3.4]. Let $(\widehat{K}, \widehat{d})$ be the completion of $(V_*, R_{p,\mathcal{E}_{p,*}}^{1/p})$ and let $(\widehat{\mathcal{E}}_{p,*}, \widehat{\mathcal{F}}_{p,*})$ be the p-resistance form on $\widehat{K}$ defined by (6.26) and (6.27), where we choose $\mathcal{S} = \{(V_n, \mathcal{E}_p^{(n)})\}_{n \in \mathbb{N} \cup \{0\}}$. Also, we fix a metric d on K which gives the original topology of K. Recall that $R_{\widehat{\mathcal{E}}_{p,*}}^{1/p} = \widehat{d}$ by Corollary 6.23. For $n \in \mathbb{N}$, we define

$$\delta_n := \min_{v,w \in W_n; K_v \cap K_w = \emptyset} \left(\inf_{x \in F_v(V_*), y \in F_w(V_*)} R_{\mathcal{E}_{p,*}}(x, y) \right).$$

Then $\delta_n > 0$ since $R_{\mathcal{E}_{p,*}}(x, y) \geq \mathcal{E}_{p,*}\bigl(h_{V_n}^{\mathcal{E}_{p,*}}[\mathbb{1}_{F_w(V_0)}]\bigr)^{-1}$ for any $x \in F_v(V_*), y \in F_w(V_*)$. Let $\{x_n\}_{n \geq 0}$ be a Cauchy sequence in $(V_*, R_{\mathcal{E}_{p,*}}^{1/p})$. For each $n \in \mathbb{N}$, choose $N(n) \in \mathbb{N}$ so that

$$\sup_{k,l \geq N(n)} R_{\mathcal{E}_{p,*}}(x_k, x_l) < \delta_n.$$

Then there exists $w \in W_n$ such that $\{x_k\}_{k \geq N(n)} \subseteq \bigcup_{v \in W_n; K_v \cap K_w \neq \emptyset} F_v(V_*) =: A_{n,w}$. Since $\lim_{n\to\infty} \max_{w \in W_n} \operatorname{diam}(A_{n,w}, d) = 0$ by (5.3), we conclude that $\mathrm{id}_{V_*} \colon (V_*, R_{\mathcal{E}_{p,*}}^{1/p}) \to (V_*, d|_{V_* \times V_*})$ is uniformly continuous. Now we define $\theta \colon (\widehat{K}, \widehat{d}) \to (K, d)$ as the unique continuous map satisfying $\theta|_{V_*} = \mathrm{id}_{V_*}$. Let us show that θ is injective. Assume that $x, y \in \widehat{K}$ satisfy $\theta(x) = \theta(y)$. Let $\{x_n\}_{n \geq 0}, \{y_n\}_{n \geq 0}$ be Cauchy sequences in $(V_*, R_{\mathcal{E}_{p,*}}^{1/p})$ satisfying $\lim_{n\to\infty} \widehat{d}(x, x_n) = \lim_{n\to\infty} \widehat{d}(y, y_n) = 0$. Then $\lim_{n\to\infty} d(\theta(x), x_n) = \lim_{n\to\infty} d(\theta(y), y_n) = 0$ since θ is continuous. For any $u \in \widehat{\mathcal{F}}_{p,*}$, let $\widehat{u}_n \in C(K)$ be the unique function satisfying $\widehat{u}_n|_{V_*} = h_{V_n}^{\mathcal{E}_{p,*}}[u|_{V_n}]$, which exists by Proposition 8.40. Also, let $v_n \in C(\widehat{K})$ be the unique function satisfying $v_n|_{V_*} = h_{V_n}^{\mathcal{E}_{p,*}}[u|_{V_n}]$; recall the proof of Theorem 6.22. Then we see that

$$v_n(x) = \lim_{k\to\infty} h_{V_n}^{\mathcal{E}_{p,*}}[u](x_k) = \widehat{u}_n(\theta(x)) = \widehat{u}_n(\theta(y)) = \lim_{k\to\infty} h_{V_n}^{\mathcal{E}_{p,*}}[u](y_k) = v_n(y). \tag{8.57}$$

Let us fix $o \in V_0 \subseteq V_n$. By (6.3) for $(\widehat{\mathcal{E}}_{p,*}, \widehat{\mathcal{F}}_{p,*})$,

$$|u(x) - v_n(x)|^p \leq R_{\widehat{\mathcal{E}}_{p,*}}(x, o)\widehat{\mathcal{E}}_{p,*}(u - \check{u}_n) = R_{\widehat{\mathcal{E}}_{p,*}}(x, o)\mathcal{E}_{p,*}\bigl(u|_{V_*} - h_{V_n}^{\mathcal{E}_{p,*}}[u|_{V_n}]\bigr),$$

which together with (6.17) and (8.57) implies that

$$u(x) = \lim_{n\to\infty} v_n(x) = \lim_{n\to\infty} v_n(y) = u(y).$$

Since $u \in \widehat{\mathcal{F}}_{p,*}$ is arbitrary, we conclude that $R_{\widehat{\mathcal{E}}_{p,*}}(x, y) = 0$ and hence $x = y$. This means that θ is injective.

Next we see that $\{F_i\}_{i \in S}$ yields a family of contractions on the complete (non-empty) metric space $(\widehat{K}, \widehat{d})$. By virtue of (8.56), similarly to the proof of (7.1), one can show that for any $w \in W_*$ and any $x, y \in V_*$,

$$\widehat{d}(F_w(x), F_w(y))^p = R_{\widehat{\mathcal{E}}_{p,*}}(F_w(x), F_w(y)) \leq \rho_{p,w}^{-1} R_{\widehat{\mathcal{E}}_{p,*}}(x, y) = \rho_{p,w}^{-1}\widehat{d}(x, y)^p.$$

In particular, $F_w|_{V_*}\colon (V_*,\widehat{d}) \to (V_*,\widehat{d})$ is uniformly continuous, and hence there exists a unique continuous map $F_w^{\widehat{K}}\colon \widehat{K} \to \widehat{K}$ such that $F_w^{\widehat{K}}\big|_{V_*} = F_w|_{V_*}$. Then it is clear that

$$\widehat{d}\bigl(F_w^{\widehat{K}}(x), F_w^{\widehat{K}}(y)\bigr) \le \rho_{p,w}^{-1/p}\,\widehat{d}(x,y) \quad \text{for any } x,y \in \widehat{K}, \tag{8.58}$$

and that $\theta \circ F_w^{\widehat{K}} = F_w \circ \theta$. Now, by (**R**) and (8.58), $\bigl\{F_i^{\widehat{K}}\bigr\}_{i\in S}$ is a family of contractions on $(\widehat{K},\widehat{d})$. By [Kig01, Theorem 1.1.4], there exists a unique non-empty compact subset $\widehat{K}_0$ of $\widehat{K}$ such that $\widehat{K}_0 = \bigcup_{i\in S} F_i^{\widehat{K}}(\widehat{K}_0)$. Let us fix $o \in \widehat{K}_0$ and set $A \coloneqq \bigcup_{w\in W_*} F_w^{\widehat{K}}(o) \subseteq \widehat{K}_0$. Then $\theta(A) = \bigcup_{w\in W_*} F_w(\theta(o))$ is dense in (K,d) by (5.3). Since $\theta(A) \subseteq \theta(\widehat{K}_0) \subseteq K$ and $\theta(\widehat{K}_0)$ is compact by the continuity of θ, we have $\theta(\widehat{K}_0) = K$ and thus $\theta(\widehat{K}) = K$. Then $\widehat{K}$ turns out to be compact since $\widehat{K} = \widehat{K}_0$ by the injectivity of θ. Now θ turns out to be a homeomorphism between $\widehat{K}$ and K. From the uniqueness of θ, we conclude that $\widehat{K} = K$ and $\theta = \mathrm{id}_K$. We complete the proof. □

The following theorem describes a construction of self-similar p-resistance form as the inductive limit of $\{\mathcal{E}_p^{(n)}\}_{n\ge 0}$ under the assumption that (**R**) holds.

Theorem 8.42. *Assume that* (**A**') *and* (**R**) *hold. We define*

$$\mathcal{F}_p \coloneqq \Bigl\{u \in C(K) \Bigm| \lim_{n\to\infty} \mathcal{E}_p^{(n)}(u|_{V_n}) < \infty\Bigr\}, \quad \textit{and} \tag{8.59}$$

$$\mathcal{E}_p(u) \coloneqq \lim_{n\to\infty} \mathcal{E}_p^{(n)}(u), \quad u \in \mathcal{F}_p. \tag{8.60}$$

Then $(\mathcal{E}_p,\mathcal{F}_p)$ is a regular self-similar p-resistance form on $\mathcal{L}$ with weight $\boldsymbol{\rho}_p$, $\mathcal{E}_p|_{V_n} = \mathcal{E}_p^{(n)}$ for any $n \in \mathbb{N}\cup\{0\}$, and $R_{\mathcal{E}_p}$ is compatible with the original topology of K.

Remark 8.43. Similar to Proposition 5.22, by choosing a suitable $E_0 \in \mathcal{S}_p(V_0)$ in Theorem 8.37, we can verify nice properties like the *symmetry-invariance* (see (9.7) for details) of E_* in (8.49), $\mathcal{E}_p^{(0)}$ in (8.50) and $\mathcal{E}_p$. See also Theorem 8.52.

Proof. By Lemma 8.41 and Corollary 6.23, $(\mathcal{E}_p,\mathcal{F}_p)$ is a p-resistance form on K. The self-similarity conditions, (5.5) and (5.6), for $(\mathcal{E}_p,\mathcal{F}_p)$ are obvious from Proposition 8.39. By Lemma 8.41 and Proposition 8.40, $R_{\mathcal{E}_p}$ is compatible with the original topology of K and $(\mathcal{E}_p,\mathcal{F}_p)$ is regular (recall Definition 6.5). □

Let us recall the following proposition, which is useful to verify (**R**) for concrete examples.

Proposition 8.44 ([CGQ22, Lemma 5.4])**.** *Assume that* (**A**') *holds. If $w \in W_*$ satisfies $\dot{w} \coloneqq www\cdots \in \mathcal{P}_{\mathcal{L}}$, then $\rho_{p,w} > 1$.*

Next we move to the elliptic Harnack inequality for (non-negative) p-harmonic functions. In the same setting as Theorem 8.42, one can verify the assumptions in Theorem 6.37 (elliptic Harnack inequality). Indeed, (8.61) and (8.62) in the proposition below are proved in [KS.a, Lemma 6.8, Propositions 6.9 and 6.14] (see also Lemma B.4-(b), (c) for affine nested fractals) and (8.63) is implied by Proposition 7.12. We summarize the results in the next proposition and corollary.

Proposition 8.45. *Assume that* (**A'**) *and* (**R**) *hold. Let* $d_{\mathrm{f}}(\boldsymbol{\rho}_p) \in (0,\infty)$ *be such that* $\sum_{i\in S}\rho_{p,i}^{-d_{\mathrm{f}}(\boldsymbol{\rho}_p)/(p-1)} = 1$, *let* m *be the self-similar measure on* $\mathcal{L}$ *with weight* $\bigl(\rho_{p,i}^{-d_{\mathrm{f}}(\boldsymbol{\rho}_p)/(p-1)}\bigr)_{i\in S}$, *let* $(\mathcal{E}_p,\mathcal{F}_p)$ *be the p-resistance form given in Theorem 8.42, and let* $\{\Gamma_{\mathcal{E}_p}\langle u\rangle\}_{u\in\mathcal{F}_p}$ *be the associated p-energy measures (recall* (5.11)*). Set* $\widehat{R}_p := \widehat{R}_{p,\mathcal{E}_p}$ *for simplicity. Then there exist* $C, A \in (1,\infty)$ *such that for any* $(x,s) \in K\times(0,\infty)$ *with* $B_{\widehat{R}_p}(x,s) \neq K$ *and any* $u \in \mathcal{F}_{p,\mathrm{loc}}(B_{\widehat{R}_p}(x,As))$,

$$C^{-1}s^{d_{\mathrm{f}}(\boldsymbol{\rho}_p)} \le m(B_{\widehat{R}_p}(x,s)) \le Cs^{d_{\mathrm{f}}(\boldsymbol{\rho}_p)}, \tag{8.61}$$

$$\inf\{\mathcal{E}_p(u) \mid u\in\mathcal{F}_p, u|_{B_{\widehat{R}_p}(x,s)} = 1, \mathrm{supp}_K[u] \subseteq B_{\widehat{R}_p}(x,As)\} \le Cs^{-(p-1)}, \tag{8.62}$$

$$\int_{B_{\widehat{R}_p}(x,s)}\left|u - \fint_{B_{\widehat{R}_p}(x,s)} u\,dm\right|^p dm \le Cs^{d_{\mathrm{f}}(\boldsymbol{\rho}_p)+p-1}\int_{B_{\widehat{R}_p}(x,As)} d\Gamma_{\mathcal{E}_p}\langle u\rangle. \tag{8.63}$$

Corollary 8.46 (Elliptic Harnack inequality on p.-c.f. self-similar structures)**.** *Assume that* (**A'**) *and* (**R**) *hold. Let* $d_{\mathrm{f}}(\boldsymbol{\rho}_p) \in (0,\infty)$, m, $(\mathcal{E}_p,\mathcal{F}_p)$ *and* $\{\Gamma_{\mathcal{E}_p}\langle u\rangle\}_{u\in\mathcal{F}_p}$ *be the same as in Proposition 8.45 Then the assumptions in Theorem 6.37 holds with* $m, d_{\mathrm{f}}(\boldsymbol{\rho}_p), d_{\mathrm{f}}(\boldsymbol{\rho}_p) + p - 1$ *in place of* μ, Q, β.

8.4 Verifying (A) for strongly symmetric p.-c.f. self-similar sets

Let us conclude this section by showing (**A**) for a special class of p.-c.f. self-similar sets called *affine nested fractals*, which was introduced in [FHK94] as a generalization of the class called nested fractals introduced by Liondstrøm [Lin90]. More precisely, we will work in a wider class called *strongly symmetric p.-c.f. self-similar sets*. The proof of (**A**) for affine nested fractals was given in [CGQ22, Theorem 6.3], but their description on the group of symmetries in the paper [CGQ22] is not sophisticated[10], so we provide the details of the proof for (**A**) and improve the assumptions in [CGQ22, Theorem 6.3] simultaneously in Theorem 8.51.

We start with recalling the definitions of a group of symmetries, affine nested fractals and strongly symmetric p.-c.f. self-similar sets. See, e.g., [Kig01, Section 3.8] for details.

Framework 8.47. Let $D \in \mathbb{N}$ and let S be a non-empty finite set with $\#S \ge 2$. Let $\{c_i\}_{i\in S} \subseteq (0,1)$, $\{a_i\}_{i\in S} \subseteq \mathbb{R}^D$ and $\{U_i\}_{i\in S} \subseteq O(D)$, where $O(D)$ is the collection of orthogonal transformations of $\mathbb{R}^D$. Define $f_i\colon \mathbb{R}^D \to \mathbb{R}^D$ by $f_i(x) := c_iU_ix + a_i$ for each $i \in S$. Let K be the self-similar set associated with $\{f_i\}_{i\in S}$, set $F_i := f_i|_K$ for each $i \in S$ and assume that $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is a p.-c.f. self-similar structure. We also assume that K is connected, $M := \#(V_0) < \infty$ and $\sum_{i=1}^{M} q_i = 0$, where $q_i \in \mathbb{R}^D$ is defined so that $V_0 = \{q_i\}_{i=1}^{M}$. Let $d\colon K\times K \to [0,\infty)$ be the Euclidean metric on K given by $d(x,y) := |x-y|$.

[10] For a group of symmetries, say $\mathcal{G}$, one of the essential properties that is needed to prove the $\mathcal{G}$-invariance of the resulted self-similar p-energy form is Proposition 8.50-(2). We have to be careful whether this property holds for $\mathcal{G}$, but this point is not handled very well.

Definition 8.48 ([Kig01, Definitions 3.8.3 and 3.8.4]). (1) We define

$$\mathcal{G}_{\mathrm{sym}}(\mathcal{L}) := \mathcal{G}_{\mathrm{sym}} := \left\{ g|_K \;\middle|\; \begin{array}{l} g \in O(D), \text{ for any } n \in \mathbb{N} \cup \{0\} \text{ and any} \\ w \in W_n \text{ there exists } w' \in W_n \text{ such that} \\ g(K_w) = K_{w'} \text{ and } g(F_w(V_0)) = F_{w'}(V_0) \end{array} \right\},$$

where $O(D)$ denotes the orthogonal group in dimension D.

(2) For $x, y \in \mathbb{R}^D$ with $x \neq y$, let $g_{xy} \colon \mathbb{R}^D \to \mathbb{R}^D$ be the reflection in the hyperplane $\{z \in \mathbb{R}^D \mid |x - z| = |y - z|\}$.

(3) Let $m_* := \#\{|x - y| \mid x, y \in V_0, x \neq y\}$ and $l_0 := \min\{|x - y| \mid x, y \in V_0, x \neq y\}$. We define $\{l_i\}_{i=0}^{m_*-1}$ inductively by $l_{i+1} := \min\{|x - y| \mid x, y \in V_0, |x - y| > l_i\}$.

(4) Let $m \in \mathbb{N} \cup \{0\}$ and $(x_i)_{i=1}^n \in (V_m)^n$. Then $(x_i)_{i=1}^n$ is called an *m-walk* (between x_1 and x_n) if and only if there exist $w^1, \dots, w^n \in W_m$ such that $\{x_i, x_{i+1}\} \subseteq F_{w^i}(V_0)$ for all $i \in \{1, 2, \dots, n-1\}$. A 0-walk $(x_i)_{i=1}^n$ is called a *strict 0-walk* (between x_1 and x_n) if and only if $|x_i - x_{i+1}| = l_0$ for any $i \in \{1, 2, \dots, n-1\}$.

(5) $\mathcal{L}$ is called a *strongly symmetric p.-c.f. self-similar set* if and only if it satisfies the following four conditions:

(SS1) For any $x, y \in V_0$ with $x \neq y$, there exists a strict 0-walk between x and y.

(SS2) If $x, y, z \in V_0$ and $|x - y| = |x - z|$, then there exists $g \in \mathcal{G}_{\mathrm{sym}}$ such that $g(x) = x$ and $g(y) = z$.

(SS3) For any $i \in \{0, \dots, m_* - 2\}$, there exist $x, y, z \in V_0$ such that $|x - y| = l_i$, $|x - z| = l_{i+1}$ and $g_{yz}|_K \in \mathcal{G}_{\mathrm{sym}}$.

(SS4) V_0 is $\mathcal{G}_{\mathrm{sym}}$-transitive, i.e., for any $x, y \in V_0$ with $x \neq y$, there exists $g \in \mathcal{G}_{\mathrm{sym}}$ such that $g(x) = y$.

(6) $\mathcal{L}$ is called an *affine nested fractal* if $g_{xy}|_K \in \mathcal{G}_{\mathrm{sym}}(\mathcal{L})$ for any $x, y \in V_0$ with $x \neq y$.

Remark 8.49. In [Kig01, Definitions 3.8.3 and 3.8.4], the following group of symmetries $\mathcal{G}_s$ is used instead of $\mathcal{G}_{\mathrm{sym}}$:

$$\mathcal{G}_s := \left\{ g|_K \;\middle|\; \begin{array}{l} g \in O(D), \text{ for any } n \in \mathbb{N} \cup \{0\} \text{ and any } w \in W_n \\ \text{there exists } w' \in W_n \text{ such that } g(F_w(V_0)) = F_{w'}(V_0) \end{array} \right\};$$

note that $\mathcal{G}_{\mathrm{sym}} \subseteq \mathcal{G}_s$. Under the assumption that

$$\#(F_i(V_0) \cap F_j(V_0)) \leq 1 \quad \text{for any } i, j \in S \text{ with } i \neq j, \tag{8.64}$$

we know that $\mathcal{G}_{\mathrm{sym}} = \mathcal{G}_s$ by [Kig01, Proposition 3.8.19]. The difference between $\mathcal{G}_{\mathrm{sym}}$ and $\mathcal{G}_s$ does not affect the arguments in the parts of [Kig01, CGQ22] (Proposition 8.50 and Theorem 8.51 below) that we need.

Let us recall a few properties of $\mathcal{G}_{\mathrm{sym}}$ and of affine nested fractals in the following proposition, which can be shown in the same ways as in [Kig01, Section 3.8]. (Let us emphasize that we do not assume (8.64) unlike [Kig01, Section 3.8].)

Proposition 8.50 ([Kig01, Propositions 3.8.7, 3.8.20 and Lemma 3.8.23]). (1) *If $\mathcal{L}$ is an affine nested fractal, then it is a strongly symmetric self-similar set.*

(2) *Let* $w \in W_*$, $g \in \mathcal{G}_{\mathrm{sym}}$ *and set*

$$U_{g,w} \coloneqq F_{w'}^{-1} \circ g \circ F_w,$$

where $w' \in W_*$ *is the unique word satisfying* $F_{w'}(V_0) = g(F_w(V_0))$. *Then* $U_{g,w} \in \mathcal{G}_{\mathrm{sym}}$.

(3) *Let* $a, b \in V_0$ *and assume that* $g_{ab}|_K \in \mathcal{G}_{\mathrm{sym}}$. *if* $x, y \in F_w(V_0)$ *for some* $w \in W_*$, $|x-b| < |x-a|$ *and* $|y-b| > |y-a|$, *then* $g_{ab}(K_w) = K_w$.

Now we can present the following theorem proving the existence of an eigenform on V_0 for strongly symmetric self-similar sets and improving [CGQ22, Theorem 6.3]. Note that the case $p = 2$ corresponds to the existence of a harmonic structure on $\mathcal{L}$ in [Kig01, Theorem 3.8.10].

Theorem 8.51. *Assume that* $\mathcal{L}$ *is strongly symmetric. If*

$$\rho_{p,i} = \rho_{p,g^{(1)}(i)} \quad \text{for any } i \in S \text{ and } g \in \mathcal{G}_{\mathrm{sym}}, \tag{8.65}$$

then $\boldsymbol{\rho}_p$ *satisfies* (**A**). *In particular, if there exists* $\rho_p \in (0,\infty)$ *such that* $\rho_{p,i} = \rho_p$ *for any* $i \in S$, *then* (**A'**) *and* (**R**) *with* $\bigl(\lambda(\boldsymbol{\rho}_p)^{-1}\rho_p\bigr)_{i \in S}$ *in place of* $\boldsymbol{\rho}_p$ *hold, where* $\lambda(\boldsymbol{\rho}_p)$ *is the number given in Theorem 8.37-(a).*

Proof. The proof is essentially the same as [CGQ22, Proof of Theorem 6.3], but we give the details of the argument since we weaken the assumption of [CGQ22, Theorem 6.3]. For $n \in \mathbb{N} \cup \{0\}$, define $E_{p,n} \in \mathcal{S}_p(V_n)$ by

$$E_{p,n}(y) \coloneqq \sum_{w \in W_n} \rho_{p,w} \sum_{x,y \in V_0; |x-y| = l_0} |u(F_w(x)) - u(F_w(y))|^p, \quad u \in \mathbb{R}^{V_n}.$$

Note that, by Proposition 8.50-(2) and (8.65), $E_{p,n}$ is $\mathcal{G}_{\mathrm{sym}}$-invariant, i.e., $E_{p,n}(u \circ g|_{V_n}) = E_{p,n}(u)$ for any $u \in \mathbb{R}^{V_n}$ and $g \in \mathcal{G}_{\mathrm{sym}}$. We fix $a_1, a_2 \in V_0$ that satisfy $|a_1 - a_2| = l_0$ and claim that for any $n \in \mathbb{N}$ and $x, y \in V_0$ with $x \neq y$,

$$\frac{1}{2} R_{E_{p,n}}(a_1, a_2) \le R_{E_{p,n}}(x,y) \le (\#V_0)^p R_{E_{p,n}}(a_1, a_2), \tag{8.66}$$

which implies (**A**) for $\boldsymbol{\rho}_p$ with $c = 2(\#V_0)^{-p}$.

We first show the upper estimate in (8.66). Let $(x_i)_{i=0}^k \in (V_0)^{k+1}$ be a strict 0-walk between x and y. Then, by (SS2), (SS4) and the $\mathcal{G}_{\mathrm{sym}}$-invariance of $E_{p,n}$, we have $R_{E_{p,n}}(x_i, x_{i+1}) = R_{E_{p,n}}(a_1, a_2)$ for any $i \in \{0, \dots, k-1\}$. Hence we see that

$$R_{E_{p,n}}(x,y)^{1/p} \le \sum_{i=0}^{k-1} R_{E_{p,n}}(x_i, x_{i+1})^{1/p} = k R_{E_{p,n}}(a_1, a_2)^{1/p} \le (\#V_0) R_{E_{p,n}}(a_1, a_2)^{1/p},$$

which shows the desired estimate.

Next we prove the lower estimate in (8.66). The case $|x-y| = l_0$ is clear by (SS2), (SS4) and the $\mathcal{G}_{\mathrm{sym}}$-invariance of $E_{p,n}$, so we assume that $|x-y| > l_0$. By (SS1), there exists $z \in V_0$ such that $|x-z| = l_0$. Define $u \in \mathbb{R}^{V_n}$ by

$$u(a) \coloneqq \begin{cases} h^{E_{p,n}}_{\{x,z\}}[\mathbb{1}_x](a) & \text{if } a \in V_n \text{ satisfies } |a-z| \leq |a-y|, \\ h^{E_{p,n}}_{\{x,z\}}[\mathbb{1}_x](g_{yz}(a)) & \text{if } a \in V_n \text{ satisfies } |a-z| \geq |a-y|, \end{cases}$$

which is well-defined since Theorem 6.13 implies $h^{E_{p,n}}_{\{x,z\}}[\mathbb{1}_x](a) = 1/2$ whenever $|a-z| = |a-y|$. Since $|x-z| = l_0 < |x-y|$, we have $u(x) = h^{E_{p,n}}_{\{x,z\}}[\mathbb{1}_x](x) = 1$. Also, $u(y) = h^{E_{p,n}}_{\{x,z\}}[\mathbb{1}_x](g_{yz}(y)) = 0$. Hence $R_{E_{p,n}}(x,y) \geq E_{p,n}(u)^{-1}$. Now we define $H_{1,n} \coloneqq \{a \in V_n \mid |a-z| \leq |a-y|\}$, $H_{2,n} \coloneqq \{a \in V_n \mid |a-z| \geq |a-y|\}$ and we see that

$$\begin{aligned} E_{p,n}(u) &= \left(\sum_{\substack{w \in W_n; \\ F_w(V_0) \subseteq H_{1,n}}} + \sum_{\substack{w \in W_n; \\ F_w(V_0) \subseteq H_{2,n}}} + \sum_{\substack{w \in W_n; \\ F_w(V_0) \not\subseteq H_{i,n}}} \right) \rho_{p,w} E_{p,0}(u \circ F_w|_{V_0}) \\ &= 2 \sum_{\substack{w \in W_n; \\ F_w(V_0) \subseteq H_{1,n}}} \rho_{p,w} E_{p,0}\Bigl(h^{E_{p,n}}_{\{x,z\}}[\mathbb{1}_x] \circ F_w|_{V_0} \Bigr) + \sum_{\substack{w \in W_n; \\ F_w(V_0) \not\subseteq H_{i,n}}} \rho_{p,w} E_{p,0}(u \circ F_w|_{V_0}). \end{aligned}$$

To estimate the second term in the right-hand side in the above inequality, let $a, b \in V_0$ satisfy $|a-b| = l_0$, $|F_w(a) - z| < |F_w(a) - y|$ and $|F_w(b) - z| > |F_w(b) - y|$. Then we have $g_{yz}(F_w(V_0)) = F_w(V_0)$ by Proposition 8.50-(3). This along with the minimality of l_0 implies that $g_{yz}(F_w(a)) = F_w(b)$, whence it follows that $u(F_w(a)) = u(F_w(b))$. Hence

$$\begin{aligned} \sum_{\substack{w \in W_n; \\ F_w(V_0) \not\subseteq H_{i,n}}} \rho_{p,w} E_{p,0}(u \circ F_w|_{V_0}) &= \sum_{\substack{w \in W_n; \\ F_w(V_0) \not\subseteq H_{i,n}}} \rho_{p,w} \sum_{\substack{a,b \in V_0; |a-b| = l_0, \\ \{F_w(a), F_w(b)\} \subseteq H_{1,n} \\ \text{or} \{F_w(a), F_w(b)\} \subseteq H_{2,n}}} |u(F_w(a)) - u(F_w(b))|^p \\ &\leq 2 \sum_{\substack{w \in W_n; \\ F_w(V_0) \not\subseteq H_{i,n}}} \rho_{p,w} E_{p,0}\Bigl(h^{E_{p,n}}_{\{x,z\}}[\mathbb{1}_x] \circ F_w|_{V_0} \Bigr), \end{aligned}$$

and we deduce that

$$R_{E_{p,n}}(x,y) \geq E_{p,n}(u)^{-1} \geq \frac{1}{2} E_{p,n}\Bigl(h^{E_{p,n}}_{\{x,z\}}[\mathbb{1}_x] \Bigr)^{-1} = \frac{1}{2} R_{E_{p,n}}(a_1, a_2),$$

completing the proof. □

The following theorem gives symmetry-invariance self-similar p-resistance forms on strongly symmetric self-similar sets.

Theorem 8.52. *Assume that $\mathcal{L}$ is a strongly symmetric p.-c.f. self-similar set and that* (**A'**), (**R**), (8.65) *hold. Then there exists a self-similar p-resistance form $(\mathcal{E}_p, \mathcal{F}_p)$ on $\mathcal{L}$ with weight $\boldsymbol{\rho}_p$ such that $u \circ g \in \mathcal{F}_p$ and $\mathcal{E}_p(u \circ g) = \mathcal{E}_p(u)$ for any $u \in \mathcal{F}_p$ and any $g \in \mathcal{G}_{\mathrm{sym}}$.*

Proof. Define $E_0 \in \mathcal{S}_p(V_0)$ by $E_0(u) \coloneqq \sum_{x,y\in V_0} |u(x) - u(y)|^p$ for $u \in \mathbb{R}^{V_0}$. Then $E_0(u) = E_0(u \circ g)$ for any $u \in \mathbb{R}^{V_0}$ and $g \in \mathcal{G}_{\mathrm{sym}}$. By Theorem 8.51 and explicit expressions (8.48), (8.49) and (8.50), there exists a p-resistance form $\mathcal{E}_p^{(0)}$ on V_0 such that $\mathcal{R}_{\boldsymbol{\rho}_p}(\mathcal{E}_p^{(0)}) = \mathcal{E}_p^{(0)}$ and $\mathcal{E}_p^{(0)}(u) = \mathcal{E}_p^{(0)}(u \circ g)$ for any $u \in \mathbb{R}^{V_0}$ and $g \in \mathcal{G}_{\mathrm{sym}}$. The desired symmetry-invariance for $(\mathcal{E}_p, \mathcal{F}_p)$ is immediate from (8.65), Proposition 8.50-(2) and the expressions (8.59), (8.60). □

9 p-Walk dimension of Sierpiński carpets/gaskets

In this section, we prove the strict inequality $d_{\mathrm{w},p} > p$ for generalized Sierpiński carpets and D-dimensional level-l Sierpiński gasket as an application of the nonlinear potential theory developed in Sections 6 and 7. In particular, we remove the *planarity* in the hypothesis of the previous result [Shi24, Theorem 2.27].

9.1 Generalized Sierpiński carpets

By following [Kaj23, Section 2], we recall the definition of generalized Sierpiński carpets.

Framework 9.1. Let $D, l \in \mathbb{N}$, $D \geq 2$, $l \geq 3$ and set $Q_0 \coloneqq [0,1]^D$. Let $S \subsetneq \{0, 1, \dots, l-1\}^D$ be non-empty, define $f_i \colon \mathbb{R}^D \to \mathbb{R}^D$ by $f_i(x) \coloneqq l^{-1}i + l^{-1}x$ for each $i \in S$ and set $Q_1 \coloneqq \bigcup_{i\in S} f_i(Q_0)$, so that $Q_1 \subsetneq Q_0$. Let K be the self-similar set associated with $\{f_i\}_{i\in S}$. Note that $K \subsetneq Q_0$. Set $F_i \coloneqq f_i|_K$ for each $i \in S$ and $\mathrm{GSC}(D, l, S) \coloneqq (K, S, \{F_i\}_{i\in S})$. Let $d \colon K \times K \to [0, \infty)$ be the Euclidean metric on K given by $d(x, y) \coloneqq |x - y|$, set $d_{\mathrm{f}} \coloneqq \log_l(\#S)$, and let m be the self-similar measure on $\mathrm{GSC}(D, l, S)$ with uniform weight $(1/\#S)_{i\in S}$.

Recall that d_{f} is the Hausdorff dimension of (K, d) and that m is a constant multiple of the d_{f}-dimensional Hausdorff measure on (K, d); see, e.g., [Kig01, Proposition 1.5.8 and Theorem 1.5.7]. Note that $d_{\mathrm{f}} < D$ by $S \subsetneq \{0, 1, \dots, l-1\}^D$.

The following definition is due to Barlow and Bass [BB99, Section 2], except that the non-diagonality condition in [BB99, Hypotheses 2.1] has been strengthened later in [BBKT] to fill a gap in [BB99, Proof of Theorem 3.19]; see [BBKT, Remark 2.10-1.] for some more details of this correction.

Definition 9.2 (Generalized Sierpiński carpet)**.** $\mathrm{GSC}(D, l, S)$ is called a *generalized Sierpiński carpet* if and only if the following four conditions are satisfied:

(GSC1) (Symmetry) $f(Q_1) = Q_1$ for any isometry f of $\mathbb{R}^D$ with $f(Q_0) = Q_0$.

(GSC2) (Connectedness) Q_1 is connected.

(GSC3) (Non-diagonality) $\mathrm{int}_{\mathbb{R}^D}\bigl(Q_1 \cap \prod_{k=1}^{D}[(i_k - \varepsilon_k)l^{-1}, (i_k + 1)l^{-1}]\bigr)$ is either empty or connected for any $(i_k)_{k=1}^D \in \mathbb{Z}^D$ and any $(\varepsilon_k)_{k=1}^D \in \{0, 1\}^D$.

(GSC4) (Borders included) $[0, 1] \times \{0\}^{D-1} \subset Q_1$.

See [BB99, Remark 2.2] for a description of the meaning of each of the four conditions (GSC1), (GSC2), (GSC3) and (GSC4) in Definition 9.2. To be precise, (GSC3) is slightly different from the formulation of the non-diagonality condition in [BBKT, Subsection 2.2], but they have been proved to be equivalent to each other in [Kaj10, Theorem 2.4]; see [Kaj10, Section 2] for some other equivalent formulations of the non-diagonality condition.

In this subsection, we assume that $\mathrm{GSC}(D,l,S) = (K, S, \{F_i\}_{i\in S})$ as introduced in Framework 9.1 is a generalized Sierpiński carpet as defined in Definition 9.2.

We next ensure the existence of a symmetry-invariant p-resistance form on $\mathrm{GSC}(D,l,S)$ for $p > \dim_{\mathrm{ARC}}(K,d)$ by applying Theorem 8.29.

Definition 9.3. We define

$$\mathcal{G}_0 \coloneqq \{ f|_K \mid f \text{ is an isometry of } \mathbb{R}^D,\ f(Q_0) = Q_0 \}, \tag{9.1}$$

which forms a finite subgroup of the group of homeomorphisms of K by virtue of (GSC1).

Corollary 9.4. *Let $p \in (\dim_{\mathrm{ARC}}(K,d), \infty)$. Then Assumption 8.25 holds with $r_* = l^{-1}$, K is p-conductively homogeneous, and there exists a regular self-similar p-resistance form $(\mathcal{E}_p, \mathcal{W}^p)$ on $\mathrm{GSC}(D,l,S)$ with weight $(\sigma_p)_{i\in S}$ such that it satisfies the conditions (a)-(d) of Theorem 8.29. In particular, $(\mathcal{E}_p, \mathcal{W}^p)$ has the following property:*

$$\text{If } u \in \mathcal{W}^p \text{ and } g \in \mathcal{G}_0 \text{ then } u \circ g \in \mathcal{W}^p \text{ and } \mathcal{E}_p(u \circ g) = \mathcal{E}_p(u). \tag{9.2}$$

Proof. Assumption 8.25 and the p-conductive homogeneity for the generalized Sierpiński carpets in the case $p \in (d_{\mathrm{ARC}}, \infty)$ follow from [Kig23, Theorem 4.13] or [Shi24, Proposition 4.5 and Theorem 4.14]. Let $(\mathcal{E}_p, \mathcal{W}^p)$ be a self-similar p-resistance form given in Theorem 8.29. Then the desired properties except for (9.2) are already proved. The symmetric-invariance (9.2) follows Theorem 8.19-(c), (8.36) and the fact that $F_i^{-1} \circ g \circ F_i \in \mathcal{G}_0$ for any $i \in S$; see also Proposition 5.22-(b). □

Recall that σ_p and $d_{\mathrm{w},p}$ are defined for any $p \in (0,\infty)$ (under Assumption 8.25). We know the following monotonicity on $d_{\mathrm{w},p}/p$ in $p \in (0,\infty)$.

Proposition 9.5. *$d_{\mathrm{w},p}/p \geq d_{\mathrm{w},q}/q$ for any $p, q \in (0,\infty)$ with $p \leq q$.*

Proof. This follows from [Kig20, Lemma 4.7.4] and the fact that $d_{\mathrm{f}} = \log_l(\#S)$. □

The following definition is exactly the same as a part of [Kaj23, Definition 3.6].

Definition 9.6. (1) We set $V_0^\varepsilon \coloneqq K \cap (\{\varepsilon\} \times \mathbb{R}^{D-1})$ for each $\varepsilon \in \{0,1\}$ and $U_0 \coloneqq K \setminus (V_0^0 \cup V_0^1)$.

(2) We define $g_\varepsilon \in \mathcal{G}_0$ by $g_\varepsilon \coloneqq \tau_\varepsilon|_K$ for each $\varepsilon = (\varepsilon_k)_{k=1}^D \in \{0,1\}^D$, where $\tau_\varepsilon \colon \mathbb{R}^D \to \mathbb{R}^D$ is given by $\tau_\varepsilon((x_k)_{k=1}^D) \coloneqq (\varepsilon_k + (1 - 2\varepsilon_k)x_k)_{k=1}^D$, and define a subgroup $\mathcal{G}_1$ of $\mathcal{G}_0$ by

$$\mathcal{G}_1 \coloneqq \{ g_\varepsilon \mid \varepsilon \in \{0\} \times \{0,1\}^{D-1} \}. \tag{9.3}$$

In the rest of this subsection, we fix $p \in (d_{\mathrm{ARC}}, \infty)$ and a self-similar p-resistance form $(\mathcal{E}_p, \mathcal{W}^p)$ in Corollary 9.4. Recall Theorem 6.13 and let $h_0 := h^{\mathcal{E}_p}_{V_0^0 \cup V_0^1}[\mathbb{1}_{V_0^1}] \in \mathcal{W}^p$. The strategy to prove $d_{\mathrm{w},p} > p$ is very similar to [Kaj23], that is, we will prove the *non-$\mathcal{E}_p$-harmonicity* on U_0 of $h_2 := \sum_{w \in W_2} (F_w)_*(l^{-2}h_0 + q_1^w \mathbb{1}_K) \in \mathcal{W}^p$, which also satisfies $h_2|_{V_0^i} = i \, (i = 0, 1)$. (See [Kaj23, Figures 2 and 3] for an illustration of h_0 and h_2.) Then the strict estimate $d_{\mathrm{w},p} > p$ will follow from $\mathcal{E}_p(h_0) < \mathcal{E}_p(h_2)$ and the self-similarity for $\mathcal{E}_p$. Our arguments will be easier than that of [Kaj23] by virtue of $\mathcal{W}^p \subseteq C(K)$.

The next proposition is a key ingredient. Note that it requires our standing assumption that $S \neq \{0, 1, \dots, l-1\}^D$, which excludes the case of $K = [0,1]^D$ from the present framework.

Proposition 9.7. *Let $h_2 := \sum_{w \in W_2} (F_w)_*(l^{-2}h_0 + q_1^w \mathbb{1}_K) \in \mathcal{W}^p$. Then h_2 is not $\mathcal{E}_p$-harmonic on U_0 and $h_2|_{V_0^i} = i$ for each $i \in \{0, 1\}$.*

Proof. The proof is a straightforward modification of [Kaj23, Proposition 3.11] thanks to Theorem 6.13. We present here a self-contained proof for the reader's convenience.

We claim that, if h_2 were $\mathcal{E}_p$-harmonic on U_0, then $h_0 \in \mathcal{W}^p$ would turn out to be $\mathcal{E}_p$-harmonic on $K \setminus V_0^0$, which would imply by combining with Proposition 6.11 that $\mathcal{E}_p(h_0) = \mathcal{E}_p(h_0; h_0) = 0$, which would be a contradiction by (RF1)$_p$ and $\mathcal{W}^p \subseteq C(K)$.

For each $\varepsilon = (\varepsilon_k)_{k=1}^D \in \{1\} \times \{0,1\}^{D-1}$, set $U^\varepsilon := K \cap \prod_{k=1}^D (\varepsilon_k - 1, \varepsilon_k + 1)$ and $K^\varepsilon := K \cap \prod_{k=1}^D [\varepsilon_k - 1/2, \varepsilon_k + 1/2]$. Fix $\varphi_\varepsilon \in \mathcal{W}^p \cap C_c(U^\varepsilon)$ so that $\varphi_\varepsilon|_{K^\varepsilon} = \mathbb{1}_{K^\varepsilon}$, which exists by (8.17), (RF1)$_p$ and (RF5)$_p$. Let $v \in \mathcal{W}^p \cap C_c(K \setminus V_0^0)$ and, taking an enumeration $\{\varepsilon^{(k)}\}_{k=1}^{2^{D-1}}$ of $\{1\} \times \{0,1\}^{D-1}$ and recalling Proposition 2.2(c), define $v_\varepsilon \in \mathcal{W}^p \cap C_c(U^\varepsilon)$ for $\varepsilon \in \{1\} \times \{0,1\}^{D-1}$ by $v_{\varepsilon^{(1)}} := v\varphi_{\varepsilon^{(1)}}$ and $v_{\varepsilon^{(k)}} := v\varphi_{\varepsilon^{(k)}} \prod_{j=1}^{k-1} (\mathbb{1}_K - \varphi_{\varepsilon^{(j)}})$ for $k \in \{2, \dots, 2^{D-1}\}$. Then $v - \sum_{\varepsilon \in \{1\} \times \{0,1\}^{D-1}} v_\varepsilon = v \prod_{\varepsilon \in \{1\} \times \{0,1\}^{D-1}} (\mathbb{1}_K - \varphi_\varepsilon) \in \mathcal{W}^p \cap C_c(U_0)$, hence $\mathcal{E}_p(h_0; v) = \sum_{\varepsilon \in \{1\} \times \{0,1\}^{D-1}} \mathcal{E}_p(h_0; v_\varepsilon)$ by Proposition 6.11 (with $Y = K \setminus U_0$). Therefore the desired $\mathcal{E}_p$-harmonicity of h_0 on $K \setminus V_0^0$ would be obtained by deducing that $\mathcal{E}(h_0; v_\varepsilon) = 0$ for any $\varepsilon \in \{1\} \times \{0,1\}^{D-1}$.

To this end, set $\varepsilon^{(0)} := (\mathbb{1}_{\{1\}}(k))_{k=1}^D$, take $i = (i_k)_{k=1}^D \in S$ with $i_1 < l - 1$ and $i + \varepsilon^{(0)} \notin S$, which exists by $\emptyset \neq S \subsetneq \{0, 1, \dots, l-1\}^D$ and (GSC1), and let $\varepsilon = (\varepsilon_k)_{k=1}^D \in \{1\} \times \{0,1\}^{D-1}$. We will choose $i^\varepsilon \in S$ with $F_{ii^\varepsilon}(\varepsilon) \in F_i(K \cap (\{1\} \times (0,1)^{D-1}))$ and assemble $v_\varepsilon \circ g_w \circ F_w^{-1}$ with a suitable $g_w \in \mathcal{G}_1$ for $w \in W_2$ with $F_{ii^\varepsilon}(\varepsilon) \in K_w$ into a function $v_{\varepsilon,2} \in \mathcal{W}^p \cap C_c(U_0)$. Specifically, set $i^{\varepsilon,\eta} := \bigl((l-1)(\mathbb{1}_{\{1\}}(k) + 1 - \varepsilon_k) + (2\varepsilon_k - 1)\eta_k\bigr)_{k=1}^D$ for each $\eta = (\eta_k)_{k=1}^D \in \{0\} \times \{0,1\}^{D-1}$ and $I^\varepsilon := \{\eta \in \{0\} \times \{0,1\}^{d-1} \mid i^{\varepsilon,\eta} \in S\}$, so that $i^\varepsilon := i^{\varepsilon,\mathbf{0}_D} \in S$ by (GSC4) and (GSC1) and hence $\mathbf{0}_D \in I^\varepsilon$. Thanks to $v_\varepsilon \in \mathcal{W}^p \cap C_c(U^\varepsilon)$ and $i + \varepsilon^{(0)} \notin S$ we can define $v_{\varepsilon,2} \in C(K)$ by setting

$$v_{\varepsilon,2}|_{K_w} := \begin{cases} v_\varepsilon \circ g_\eta \circ F_w^{-1} & \text{if } \eta \in I^\varepsilon \text{ and } w = ii^{\varepsilon,\eta} \\ 0 & \text{if } w \notin \{ii^{\varepsilon,\eta} \mid \eta \in I^\varepsilon\} \end{cases} \quad \text{for each } w \in W_2. \tag{9.4}$$

Then $\mathrm{supp}_K[v_{\varepsilon,2}] \subset K_i \setminus V_0^0 \subset U_0$ by (9.4) and $i_1 < l - 1$. In addition, $v_{\varepsilon,2} \circ F_w \in \mathcal{W}^p$ for any $w \in W_2$ by (9.4), $v_\varepsilon \in \mathcal{W}^p$ and (9.2). Thus $v_{\varepsilon,2} \in \mathcal{F}_p$ by (5.5) and therefore

$v_{\varepsilon,2} \in \mathcal{W}^p \cap C_c(U_0)$. Recall that $h_2 \circ F_w = l^{-2}h_0 + q_1^w \mathbb{1}_K$ for any $w \in W_2$ and note that, by the uniqueness in Theorem 6.13, $h_0 \circ g_\eta = h_0$ for any $\eta \in I^\varepsilon$. Then we have

$$\begin{aligned}\mathcal{E}_p(h_2; v_{\varepsilon,2}) &= \sum_{\eta \in I^\varepsilon} \sigma_p^2 l^{-2(p-1)} \mathcal{E}_p(h_0; v_\varepsilon \circ g_\eta) \\ &= \sum_{\eta \in I^\varepsilon} \sigma_p^2 l^{-2(p-1)} \mathcal{E}_p(h_0 \circ g_\eta; v_\varepsilon) = (\#I^\varepsilon)\sigma_p^2 l^{-2(p-1)} \mathcal{E}_p(h_0; v_\varepsilon). \end{aligned} \tag{9.5}$$

Now, supposing that h_2 were $\mathcal{E}_p$-harmonic on U_0, from (9.5), $\#I^\varepsilon > 0$, $v_{\varepsilon,2} \in \mathcal{F}_p \cap C_c(U_0)$ and Proposition 6.11, we would obtain $\mathcal{E}_p(h_0; v_\varepsilon) = \sigma_p^{-2} l^{2(p-1)} (\#I^\varepsilon)^{-1} \mathcal{E}_p(h_2; v_{\varepsilon,2}) = 0$, which would imply a contradiction as explained in the last two paragraphs. □

Theorem 9.8. *$d_{\mathrm{w},p} > p$ for any $p \in (0,\infty)$.*

Proof. It suffices to prove the case $p \in (d_{\mathrm{ARC}}, \infty)$ by Proposition 9.5. Let $h_0, h_2 \in \mathcal{W}^p$ be as in Proposition 9.7. By Proposition 9.7, we have $\mathcal{E}_p(h_0) < \mathcal{E}_p(h_2)$. This strict inequality combined with (5.6) shows that

$$\mathcal{E}_p(h_0) < \mathcal{E}_p(h_2) = \bigl(\sigma_p(\#S) l^{-p}\bigr)^2 \mathcal{E}_p(h_0),$$

whence $l^p < \sigma_p(\#S)$. Since $\sigma_p = l^{d_{\mathrm{w},p} - d_{\mathrm{f}}}$ and $d_{\mathrm{f}} = \log \#S / \log l$, we get $d_{\mathrm{w},p} = \log\bigl(\sigma_p(\#S)\bigr)/\log l > p$. □

9.2 D-dimensional level-l Sierpiński gaskets

Following [Kaj13, Example 5.1], we introduce D-dimensional level-l Sierpiński gaskets.

Framework 9.9 (D-dimensional level-l Sierpiński gaskets)**.** Let $D, l \in \mathbb{N}$, $D \geq 2$, $l \geq 2$ and let $\{q_k\}_{k=0}^D \subseteq \mathbb{R}^D$ be the set of the vertices of a regular D-dimensional simplex so that $q_0, \ldots, q_{D-1} \in \{(x_1, \ldots, x_D) \in \mathbb{R}^D \mid x_1 = 0\}$ and $q_D \in \{(x_1, \ldots, x_D) \in \mathbb{R}^D \mid x_1 \geq 0\}$. Further let $S := \bigl\{(i_k)_{k=1}^D \bigm| i_k \in \mathbb{N} \cup \{0\}, \sum_{k=1}^D i_k \leq l-1\bigr\}$, and for each $i = (i_k)_{k=1}^D \in S$ we set $q_i := q_0 + \sum_{k=1}^D l^{-1} i_k (q_k - q_0)$ and define $f_i \colon \mathbb{R}^D \to \mathbb{R}^D$ by $f_i(x) := q_i + l^{-1}(x - q_0)$. Let K be the self-similar set associated with $\{f_i\}_{i\in S}$ and set $F_i := f_i|_K$. Let $\mathrm{SG}(D,l,S) = (K, S, \{F_i\}_{i\in S})$, which is a self-similar structure. Let $d \colon K \times K \to [0,\infty)$ be the Euclidean metric on K, set $d_{\mathrm{f}} := \log_l \#S$, and let m be the self-similar measure on $\mathrm{SG}(D,l,S)$ with uniform weight $(1/\#S)_{i\in S}$.

Each $\mathrm{SG}(D,l,S)$ is called the *D-dimensional level-l Sierpiński gasket* and belongs to a class called the *nested fractals* (see [Kig01, Section 3.8] for details on nested fractals). In the rest of this subsection, we fix such a Sierpiński gasket $\mathrm{SG}(D,l,S)$ and the self-similar measure m as in Framework 9.9. We can easily verify [Kig23, Assumption 2.15] for $\mathrm{SG}(D,l,S)$. In addition, it is well known that m is d_{f}-Ahlfors regular (see [Kig23, Proposition E.7] for example). Similar to Corollary 9.4, we have a symmetry-invariant p-resistance form on $\mathrm{SG}(D,l,S)$ for any $p \in (1,\infty)$. (The Ahlfors regular conformal dimension of (K,d) is 1. See Theorem B.9.)

Definition 9.10. We define

$$\mathcal{G}_0 \coloneqq \{f|_K \mid f \text{ is an isometry of } \mathbb{R}^D,\ f(V_0) = V_0\}, \tag{9.6}$$

which forms a finite subgroup of the group of homeomorphisms of K.

Corollary 9.11. *Let $p \in (1,\infty)$. Then Assumption 8.25 holds with $r_* = l^{-1}$, K is p-conductively homogeneous, and there exists a regular self-similar p-resistance form $(\mathcal{E}_p, \mathcal{W}^p)$ on $\mathrm{SG}(D,l,S)$ with weight $(\sigma_p)_{i\in S}$ such that it satisfies the conditions (a)-(d) in Theorem 8.29. In particular, $(\mathcal{E}_p, \mathcal{W}^p)$ has the following property:*

$$\text{If } u \in \mathcal{W}^p \text{ and } g \in \mathcal{G}_0 \text{ then } u \circ g \in \mathcal{W}^p \text{ and } \mathcal{E}_p(u \circ g) = \mathcal{E}_p(u). \tag{9.7}$$

Similar to Proposition 9.5, we have the following monotonicity of $d_{\mathrm{w},p}/p$ in p.

Proposition 9.12. *$d_{\mathrm{w},p}/p \geq d_{\mathrm{w},q}/q$ for any $p, q \in (0,\infty)$ with $p \leq q$.*

We can prove the following main result by using compatible sequences.

Theorem 9.13. *$d_{\mathrm{w},p} > p$ for any $p \in (0,\infty)$.*

Proof. Let $p \in (1,\infty)$ and let $(\mathcal{E}_p, \mathcal{W}^p)$ be a self-similar p-resistance form as given in Corollary 9.11. Define $u \in C(K)$ by $u(x_1,\dots,x_D) \coloneqq x_1$ for any $(x_1,\dots,x_D) \in K \subseteq \mathbb{R}^D$. Then $u|_{V_n} \in \mathcal{W}^p|_{V_n}$ for any $n \in \mathbb{N} \cup \{0\}$ by Proposition 6.8. We claim that if $u|_{V_1}$ were $\mathcal{E}_p|_{V_1}$-harmonic on $V_1 \setminus V_0$, then $\mathcal{E}_p|_{V_0}(u|_{V_0}) = 0$, which would contradict (RF1)$_p$.

Suppose that $\mathcal{E}_p|_{V_1}(u|_{V_1};\varphi) = 0$ for every $\varphi \in \mathbb{R}^{V_1}$ with $\varphi|_{V_0} = 0$. Noting that $(u|_{V_1} \circ F_i)|_{V_0} = l^{-1}u|_{V_0} + c_i \mathbb{1}_{V_0}$ for some constant $c_i \in \mathbb{R}$ and using (7.5), we have

$$\mathcal{E}_p|_{V_1}(u|_{V_1};\varphi) = \sigma_p \sum_{i\in S} \mathcal{E}_p|_{V_0}(u|_{V_1} \circ F_i; \varphi \circ F_i) = l^{-(p-1)}\sigma_p \sum_{i\in S} \mathcal{E}_p|_{V_0}(u|_{V_0}; \varphi \circ F_i). \tag{9.8}$$

It is easy to see that $(V_1 \setminus V_0) \cap \{(x_1,\dots,x_D) \in \mathbb{R}^D \mid x_1 = 0\} \neq \emptyset$. Let $z \in V_1 \setminus V_0$ with $z \in \{x_1 = 0\}$ and let $\varphi \coloneqq \mathbb{1}_{\{z\}}^{V_1} \in \mathbb{R}^{V_1}$. Since $u \circ g = u$ for any $g \in \mathcal{G}_0$ with $g(\{x_1 = 0\} \cap K) = \{x_1 = 0\} \cap K$, the $\mathcal{G}_0$-invariance (9.7) implies $\mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}_{\{q_i\}}^{V_0}\bigr) = \mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}_{\{q_j\}}^{V_0}\bigr)$ for any $i, j \in \{0,\dots,D-1\}$. Since $\varphi \circ F_j = 0 \in \mathbb{R}^{V_0}$ for any $j \in S$ with $z \notin K_j$, we have from (9.8) that

$$\begin{aligned} 0 = \mathcal{E}_p|_{V_1}(u|_{V_1};\varphi) &= l^{-(p-1)}\sigma_p \sum_{i\in S; z\in K_i} \mathcal{E}_p|_{V_0}(u|_{V_0}; \varphi \circ F_i) \\ &= l^{-(p-1)}\sigma_p \cdot \bigl(\#\{i \in S \mid z \in K_i\}\bigr)\mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}_{\{q_0\}}^{V_0}\bigr). \end{aligned}$$

Hence we get $\mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}_{\{q_j\}}^{V_0}\bigr) = 0$ for every $j \in \{0,\dots,D-1\}$. Therefore,

$$\mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}_{\{q_D\}}^{V_0}\bigr) = \mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}_{V_0}\bigr) = \sum_{j=0}^{D-1} \mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}_{\{q_j\}}^{V_0}\bigr) = 0,$$

which yields $\mathcal{E}_p|_{V_0}(u|_{V_0}; v) = 0$ for every $v \in \mathbb{R}^{V_0}$. In particular, $\mathcal{E}_p|_{V_0}(u|_{V_0}) = 0$, which is a contradiction and hence we conclude that $u|_{V_1}$ is *not* $\mathcal{E}_p|_{V_1}$-harmonic on $V_1 \setminus V_0$. Combining with Proposition 6.15, we see that

$$\mathcal{E}_p|_{V_0}(u|_{V_0}) = \mathcal{E}_p|_{V_1}|_{V_0}(u|_{V_0}) = \mathcal{E}_p|_{V_1}\left(h_{V_0}^{\mathcal{E}_p|_{V_1}}[u|_{V0}]\right) < \mathcal{E}_p|_{V_1}(u|_{V_1}). \tag{9.9}$$

Similar to (9.8), we have $\mathcal{E}_p|_{V_1}(u|_{V_1}) = l^{-p}\sigma_p(\#S)\mathcal{E}_p|_{V_0}(u|_{V_0})$. Hence the strict inequality (9.9) yields $1 < l^{-p}l^{d_{\mathrm{w},p}-d_{\mathrm{f}}}(\#S) = l^{d_{\mathrm{w},p}-p}$, which proves $d_{\mathrm{w},p} > p$ for any $p \in (1, \infty)$. By Proposition 9.12, we complete the proof. □

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A Symmetric Dirichlet forms and the generalized contraction properties

In this section, we verify generalized contraction properties for some energy forms related with symmetric Dirichlet forms.

Throughout this section, we fix a measure space $(X, \mathcal{B}, m)$.

A.1 Symmetric Dirichlet forms satisfy the generalized 2-contraction property

In this subsection, we verify that any symmetric Dirichlet form satisfies $(\mathrm{GC})_2$.

Let us recall the definition of symmetric Dirichlet form. See, e.g., [CF, FOT, MR] for details on the theory of (symmetric) Dirichlet forms.

Definition A.1 (Symmetric Dirichlet form)**.** Let $\mathcal{F}$ be a dense linear subspace of $L^2(X, m)$ and let $\mathcal{E} \colon \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ be a non-negative definite symmetric bilinear form on $\mathcal{F}$. The pair $(\mathcal{E}, \mathcal{F})$ is said to be a *symmetric Dirichlet form* on $L^2(X, m)$ if and only if $\mathcal{F}$ equipped with the inner product $\mathcal{E} + \langle\,\cdot\,,\,\cdot\,\rangle_{L^2(X,m)}$ is a Hilbert space and $u^+ \wedge 1 \in \mathcal{F}$, $\mathcal{E}(u^+ \wedge 1, , u^+ \wedge 1) \leq \mathcal{E}(u, u)$ for any $u \in \mathcal{F}$.

We can show that a symmetric Dirichlet form $(\mathcal{E},\mathcal{F})$ satisfies $(\mathrm{GC})_2$ by modifying the proof of [MR, Theorem I.4.12].

Proposition A.2. *Let $(\mathcal{E},\mathcal{F})$ be a symmetric Dirichlet form on $L^2(X,m)$. Then $(\mathcal{E},\mathcal{F})$ is a 2-energy form on $L^2(X,m)$ satisfying $(\mathrm{GC})_2$.*

Proof. The triangle inequality for $\mathcal{E}^{1/2}$ is clear, so we shall prove $(\mathrm{GC})_2$ for $(\mathcal{E},\mathcal{F})$. Let us fix $n_1,n_2\in\mathbb{N}$, $q_1\in(0,2]$, $q_2\in[2,\infty]$ and $T=(T_1,\dots,T_{n_2})\colon\mathbb{R}^{n_1}\to\mathbb{R}^{n_2}$ satisfying (2.1) with 2 in place of p. We consider the case $q_2<\infty$ (the case $q_2=\infty$ is similar). Let $\{G_\alpha\}_{\alpha>0}$ be the strongly continuous resolvent on $L^2(X,m)$ associated with $(\mathcal{E},\mathcal{F})$; see, e.g., [MR, Theorem I.2.8]. By [MR, Theorem I.2.13-(ii)], it suffices to prove that for any $\boldsymbol{u}=(u_1,\dots,u_{n_1})\in L^2(X,m)^{n_1}$ and any $\alpha\in(0,\infty)$,

$$\left(\sum_{l=1}^{n_2}\langle(1-\alpha G_\alpha)T_l(\boldsymbol{u}),T_l(\boldsymbol{u})\rangle_{L^2(X,m)}^{q_2/2}\right)^{1/q_2}\le\left(\sum_{k=1}^{n_1}\langle(1-\alpha G_\alpha)u_k,u_k\rangle_{L^2(X,m)}^{q_1/2}\right)^{1/q_1}. \tag{A.1}$$

By the linearity of G_α and (2.1), it is enough to prove (A.1) in the case where u_k is a simple function for each $k\in\{1,\dots,n_1\}$, so we assume that

$$u_k=\sum_{i=1}^{N}\alpha_{ki}\mathbb{1}_{A_i},\quad k\in\{1,\dots,n_1\}, \tag{A.2}$$

where $N\in\mathbb{N}$, $(\alpha_{ki})_{i=1}^N\subseteq\mathbb{R}$, $\{A_i\}_{i=1}^N\subseteq\mathcal{B}(X)$ with $m(A_i)<\infty$ and $A_i\cap A_j=\emptyset$ for $i\neq j$. Fix $\alpha\in(0,\infty)$ and, for $i,j\in\{1,\dots,N\}$, we define

$$b_{i,j}\coloneqq\langle(1-\alpha G_\alpha)\mathbb{1}_{A_i},\mathbb{1}_{A_j}\rangle_{L^2(X,m)},\quad \lambda_i\coloneqq m(A_i)\quad\text{and}\quad a_{ij}\coloneqq\langle\alpha G_\alpha\mathbb{1}_{A_i},\mathbb{1}_{A_j}\rangle_{L^2(X,m)}.$$

Then $b_{ij}=\lambda_i\delta_{ij}-a_{ij}$ by a simple calculation, and $a_{ij}=a_{ji}$ since G_α is a symmetric operator on $L^2(X,m)$ (see, e.g., [MR, Theorem I.2.8]). Hence for any $(z_1,\dots,z_N)\in\mathbb{R}^N$,

$$\sum_{i,j=1}^{N}z_iz_jb_{ij}=\sum_{i<j}a_{ij}(z_i-z_j)^2+\sum_{j=1}^{N}m_jz_j^2, \tag{A.3}$$

where $m_j\coloneqq\lambda_j-\sum_{i=1}^N a_{ij}$. Note that $a_{ij}\ge 0$ for any $i,j\in\{1,\dots,N\}$ since $\alpha G_\alpha\mathbb{1}_{A_i}\ge 0$ by [MR, Theorem I.4.4]. We set $A\coloneqq\bigcup_{i=1}^N A_i$, and then we have $\alpha G_\alpha(\mathbb{1}_A)\le 1$ by [MR, Theorem I.4.4] and

$$\sum_{u=1}^{N}a_{ij}=\alpha\int_X\mathbb{1}_AG_\alpha(\mathbb{1}_{A_j})\,dm=\alpha\int_X G_\alpha(\mathbb{1}_A)\mathbb{1}_{A_j}\,dm\le\int_X\mathbb{1}_{A_j}\,dm=\lambda_j,$$

whence $m_j\ge 0$. With these preparations, we show (A.1) for $\boldsymbol{u}$ defined in (A.2). Set $T_{l,i}\coloneqq T_l(\alpha_{1i},\dots,\alpha_{u_1i})$ for each $l\in\{1,\dots,n_2\}$.

$$\sum_{l=1}^{n_2}\langle(1-\alpha G_\alpha)T_l(\boldsymbol{u}),T_l(\boldsymbol{u})\rangle_{L^2(X,m)}^{q_2/2}=\sum_{l=1}^{n_2}\left(\sum_{i,j=1}^{N}T_{l,i}T_{l,j}b_{ij}\right)^{q_2/2}$$

$$
\begin{aligned}
&\overset{(A.3)}{=} \sum_{l=1}^{n_2} \left(\sum_{i<j} a_{ij} (T_{l,i} - T_{l,j})^{q_2 \cdot \frac{2}{q_2}} + \sum_{j=1}^{N} m_j T_{l,j}^{q_2 \cdot \frac{2}{q_2}} \right)^{q_2/2} \\
&\overset{(2.19)}{\leq} \left(\sum_{i<j} \left(a_{ij}^{q_2/2} \sum_{l=1}^{n_2} (T_{l,i} - T_{l,j})^{q_2} \right)^{2/q_2} + \sum_{j=1}^{N} \left(m_j^{q_2/2} \sum_{l=1}^{n_2} T_{l,j}^{q_2} \right)^{2/q_2} \right)^{q_2/2} \\
&\overset{(2.1)}{\leq} \left(\sum_{i<j} \left(a_{ij}^{q_2/2} \left(\sum_{k=1}^{n_1} (\alpha_{ki} - \alpha_{kj})^{q_1} \right)^{q_2/q_1} \right)^{2/q_2} + \sum_{j=1}^{N} \left(m_j^{q_2/2} \left(\sum_{k=1}^{n_1} \alpha_{kj}^{q_1} \right)^{q_2/q_1} \right)^{2/q_2} \right)^{q_2/2} \\
&= \left(\sum_{i<j} \left(\sum_{k=1}^{n_1} \left(a_{ij} (\alpha_{ki} - \alpha_{kj})^2 \right)^{q_1/2} \right)^{2/q_1} + \sum_{j=1}^{N} \left(\sum_{k=1}^{n_1} (m_j \alpha_{kj}^2)^{q_1/2} \right)^{2/q_1} \right)^{\frac{q_1}{2} \cdot \frac{q_2}{q_1}} \\
&\overset{(*)}{\leq} \left(\left(\sum_{k=1}^{n_1} \left(\sum_{i<j} a_{ij} (\alpha_{ki} - \alpha_{kj})^2 + \sum_{j=1}^{N} m_j \alpha_{kj}^2 \right)^{q_1/2} \right)^{2/q_1} \right)^{\frac{q_1}{2} \cdot \frac{q_2}{q_1}} \\
&= \left(\sum_{k=1}^{n_1} \left(\sum_{i<j} a_{ij} (\alpha_{ki} - \alpha_{kj})^2 + \sum_{j=1}^{N} m_j \alpha_{kj}^2 \right)^{q_1/2} \right)^{q_2/q_1} \\
&\overset{(A.3)}{=} \left(\sum_{k=1}^{n_1} \left(\sum_{i,=1}^{N} \alpha_{ki} \alpha_{kj} b_{ij} \right)^{q_1/2} \right)^{q_2/q_1} = \left(\sum_{k=1}^{n_1} \langle (1 - \alpha G_\alpha) u_k, u_k \rangle_{L^2(X,m)}^{q_1/2} \right)^{\frac{2}{q_1} \cdot \frac{q_2}{2}},
\end{aligned}
$$

where we used the triangle inequality for ℓ^{2/q_1}-norm in $(*)$. The proof is completed. □

Next we will extend $(\mathrm{GC})_2$ to $(\mathcal{E}, \mathcal{F}_e)$, where $\mathcal{F}_e$ is the *extended Dirichlet space*; see Definition A.4 below. (See, e.g., [FOT, Section 1.5] or [CF, Section 1.1] for details on the extended Dirichlet space.) We need to recall the following result.

Proposition A.3 ([Sch99b, Proposition 1] and [Sch99a, Lemma 1][11])**.** *Assume that m is σ-finite. Let $(\mathcal{E}, \mathcal{F})$ be a symmetric Dirichlet form on $L^2(X, m)$. If $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ converges m-a.e. to 0 and $\lim_{k\wedge l\to\infty} \mathcal{E}(u_k - u_l, u_k - u_l) = 0$, then $\lim_{n\to\infty} \mathcal{E}(u_n, u_n) = 0$.*

Now we define the extended form $(\mathcal{E}, \mathcal{F}_e)$.

Definition A.4 (Extended form)**.** Let $(\mathcal{E}, \mathcal{F})$ be a symmetric Dirichlet form on $L^2(X, m)$. We define the *extended form* $(\mathcal{E}, \mathcal{F}_e)$ by

$$
\mathcal{F}_e \coloneqq \left\{ f \in L^0(X, m) \;\middle|\; \begin{array}{l} \lim_{n\to\infty} f_n = f \ m\text{-a.e. for some } \{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F} \\ \text{with } \lim_{k\wedge l\to\infty} \mathcal{E}(f_k - f_l, f_k - f_l) = 0 \end{array} \right\}, \tag{A.4}
$$

[11]More precisely, this is a special case of [Sch99a, Lemma 1]. In [Sch99a, Lemma 1], $(\mathcal{E}, \mathcal{F})$ is assumed to be a positive semi-definite bilinear form satisfying the *strong sector condition* (see [Sch99a, Definition 1]) and the Fatou property (see [Sch99a, Definition 2]), both of which are satisfies if $(\mathcal{E}, \mathcal{F})$ is a symmetric Dirichlet form. Indeed, the strong sector condition is immediate from the Cauchy–Schwarz inequality for $\mathcal{E}$ and the Fatou property for $(\mathcal{E}, \mathcal{F})$ follows from [Sch99b, Proposition 1].

$$\mathcal{E}(f, f) := \lim_{n\to\infty} \mathcal{E}(f_n, f_n), \tag{A.5}$$

where $\{f_n\}_{n\in\mathbb{N}}$ is a sequence as in (A.4). Such $\{f_n\}_{n\in\mathbb{N}}$ as in (A.4) is called an *approximating sequence for f*. (By virtue of Proposition A.3, $\lim_{n\to\infty} \mathcal{E}(f_n, f_n)$ does not depend on a particular choice of $\{f_n\}_{n\in\mathbb{N}}$. See also [FOT, Theorem 1.5.2-(i)].)

We also need the following proposition, which is proved by utilizing a version [CF, Theorem A.4.1-(ii)] of the Banach–Saks theorem .

Proposition A.5 ([Sch99a, Lemma 2][12]**).** *Assume that m is σ-finite. Let $(\mathcal{E}, \mathcal{F})$ be a symmetric Dirichlet form on $L^2(X, m)$. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$. If $\liminf_{n\to\infty} \mathcal{E}(u_n, u_n) < \infty$ and $\{u_n\}_{n\in\mathbb{N}}$ converges m-a.e. to $u \in L^0(X, m)$ as $n \to \infty$, then $u \in \mathcal{F}_e$ and $\mathcal{E}(u, u) \leq \liminf_{n\to\infty} \mathcal{E}(u_n, u_n)$.*

Now we can show that the extended form $(\mathcal{E}, \mathcal{F}_e)$ satisfies $(\mathrm{GC})_2$ under the extra assumption that m is σ-finite.

Proposition A.6. *Assume that m is σ-finite. Let $(\mathcal{E}, \mathcal{F})$ be a symmetric Dirichlet form on $L^2(X, m)$. Then $(\mathcal{E}, \mathcal{F}_e)$ is a 2-energy form on (X, m) satisfying $(\mathrm{GC})_2$.*

Proof. Set $\mathcal{E}(u) := \mathcal{E}(u, u)$ for $u \in \mathcal{F}_e$. Then $\mathcal{E}\colon \mathcal{F}_e \to [0, \infty)$ is clearly 2-homogeneous. Let us show $(\mathrm{GC})_2$ for $(\mathcal{E}, \mathcal{F}_e)$. As in the proof of Proposition A.2, let us fix $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0, 2]$, $q_2 \in [2, \infty]$ and $T = (T_1, \dots, T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfying (2.1) with 2 in place of p. Let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}_e^{n_1}$. For each $k \in \{1, \dots, n_1\}$, let $\{u_{k,n}\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ be an approximating sequence for u_k. Set $\boldsymbol{u}_n := (u_{1,n}, \dots, u_{n_1,n})$. Since $T_l \in C(\mathbb{R}^{n_1})$ and $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{GC})_2$, $\lim_{n\to\infty} T_l(\boldsymbol{u}_n) = T_l(\boldsymbol{u})$ m-a.e. and $\{\mathcal{E}(T_l(\boldsymbol{u}_n))\}_{n\in\mathbb{N}}$ is bounded. Then we have $T_l(\boldsymbol{u}) \in \mathcal{F}_e$ and $\mathcal{E}(T_l(\boldsymbol{u})) \leq \liminf_{n\to\infty} \mathcal{E}(T_l(\boldsymbol{u}_n))$ by Proposition A.5. In addition, by $(\mathrm{GC})_2$ for $(\mathcal{E}, \mathcal{F})$,

$$\begin{aligned}
\left\| \bigl(\mathcal{E}(T_l(\boldsymbol{u}))^{1/2}\bigr)_{l=1}^{n_2} \right\|_{\ell^{q_2}} &\leq \left\| \Bigl(\liminf_{n\to\infty} \mathcal{E}(T_l(\boldsymbol{u}_n))^{1/2}\Bigr)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\
&\leq \liminf_{n\to\infty} \left\| \bigl(\mathcal{E}(T_l(\boldsymbol{u}_n))^{1/2}\bigr)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\
&\leq \liminf_{n\to\infty} \left\| \bigl(\mathcal{E}(u_{k,n})^{1/2}\bigr)_{k=1}^{n_1} \right\|_{\ell^{q_1}} = \left\| \bigl(\mathcal{E}(u_k)^{1/2}\bigr)_{k=1}^{n_1} \right\|_{\ell^{q_1}},
\end{aligned}$$

which means that $(\mathcal{E}, \mathcal{F}_e)$ satisfies $(\mathrm{GC})_2$. □

A.2 The generalized contraction properties for energy measures

In this subsection, under additional topological assumptions on (X, m), we verify $(\mathrm{GC})_2$ for the (2-)energy measures associated with a regular symmetric Dirichlet form.

In the rest of this section, we assume that (X, m) satisfies (3.27), (3.28) and that X is separable, and let $\mathcal{B} = \mathcal{B}(X)$. (These are the same topological assumptions as in [FOT, (1.1.7)].)

---

[12] Similar to Proposition A.3, this proposition is true for any positive semi-definite bilinear form $(\mathcal{E}, \mathcal{F})$ satisfying the strong sector condition and the Fatou property.

Recall that $(\mathcal{E}, \mathcal{F})$ is said to be *regular* if and only if $(\mathcal{E}, \mathcal{F})$ possess a core in the sense of Definition 3.24. A regular symmetric Dirichlet form is known to satisfy the following representation.

Theorem A.7 (Beurling–Deny expression of a regular symmetric Dirichlet form; see, e.g. [FOT, Theorem 3.2.1])**.** *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular symmetric Dirichlet form on $L^2(X, m)$. Then there exist a symmetric bilinear form $\mathcal{E}^{(c)}$ on $\mathcal{F} \cap C_c(X)$ satisfying $\mathcal{E}^{(c)}(u, v) = 0$ for any $u, v \in \mathcal{F} \cap C_c(X)$ with v constant on a neighborhood of $\operatorname{supp}_X[u]$, symmetric positive Radon measure J on $X \times X$ with $J(\{(x, x) \mid x \in X\}) = 0$ and a positive Radon measure k on X such that*

$$\mathcal{E}(u, v) = \mathcal{E}^{(c)}(u, v) + \mathcal{E}^{(j)}(u, v) + \mathcal{E}^{(k)}(u, v) \quad \text{for any } u, v \in \mathcal{F} \cap C_c(X), \tag{A.6}$$

where

$$\mathcal{E}^{(j)}(u, v) \coloneqq \int_{X \times X} (u(x) - u(y))(v(x) - v(y))\, J(dx, dy), \quad \mathcal{E}^{(k)}(u, v) \coloneqq \int_X u(x)v(x)\, k(dx).$$

In addition, such $\mathcal{E}^{(c)}$, J and k are uniquely determined by $\mathcal{E}$. We call $\mathcal{E}^{(c)}$ the local part *of $\mathcal{E}$, J the* jumping measure *associated with $\mathcal{E}$ and k the* killing measure *associated with $\mathcal{E}$.*

In the next propositions, we extend each part in the decomposition (A.6) to $\mathcal{F}_e$ and associate energy measures to them. See [FOT, Chapters 2 and 3] for their proofs.

Proposition A.8. *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular symmetric Dirichlet form on $L^2(X, m)$. Let $u \in \mathcal{F}_e$ and $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}$ be an approximating sequence for u. Then, for any $\mathcal{E}^{\#} \in \{\mathcal{E}^{(c)}, \mathcal{E}^{(j)}, \mathcal{E}^{(k)}\}$, $\{\mathcal{E}^{\#}(u_n, u_n)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $[0, \infty)$ and the limit $\lim_{n \to \infty} \mathcal{E}^{\#}(u_n, u_n)$ does not depend on a particular choice of an approximating sequence $\{u_n\}_n$ for u.*

Proposition A.9. *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular symmetric Dirichlet form on $L^2(X, m)$ and let $\mathcal{E}^{\#} \in \{\mathcal{E}, \mathcal{E}^{(c)}, \mathcal{E}^{(j)}, \mathcal{E}^{(k)}\}$. For any $u \in \mathcal{F} \cap C_c(X)$, there exists a unique positive Radon measure $\mu^{\#}_{\langle u \rangle}$ on X such that*

$$\int_X \varphi\, d\mu^{\#}_{\langle u \rangle} = \mathcal{E}^{\#}(u, u\varphi) - \frac{1}{2}\mathcal{E}^{\#}(u^2, \varphi) \quad \text{for any } \varphi \in \mathcal{F} \cap C_c(X). \tag{A.7}$$

Moreover, for any Borel measurable function $\varphi \colon X \to [0, \infty)$ with $\|\varphi\|_{\sup} < \infty$, any $u \in \mathcal{F}_e$ and any approximating sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F} \cap C_c(X)$ for u, $\bigl\{\int_X \varphi\, d\mu^{\#}_{\langle u_n \rangle}\bigr\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $[0, \infty)$, $\lim_{n \to \infty} \int_X \varphi\, d\mu^{\#}_{\langle u_n \rangle}$ does not depend on the choice of $\{u_n\}_n$, and $\int_X \varphi\, d\mu^{\#}_{\langle u \rangle} = \lim_{n \to \infty} \int_X \varphi\, d\mu^{\#}_{\langle u_n \rangle}$, where $\mu^{\#}_{\langle u_n \rangle}$ is the positive Radon measure on X defined by $\mu^{\#}_{\langle u \rangle}(A) \coloneqq \lim_{n \to \infty} \mu^{\#}_{\langle u_n \rangle}(A)$ for $A \in \mathcal{B}(X)$.

Definition A.10 (Energy measures)**.** Let $u \in \mathcal{F}_e$. Let $\mu_{\langle u \rangle}$ denote the measure in the above proposition in the case $\mathcal{E}^{\#} = \mathcal{E}$. We call $\mu_{\langle u \rangle}$ the *energy measure* of u. For each $w \in \{c, j, k\}$, let $\mu^{w}_{\langle u \rangle}$ denote the measure in the above proposition in the case $\mathcal{E}^{\#} = \mathcal{E}^{(w)}$. For $u, v \in \mathcal{F}_e$, we also define $\mu^{\#}_{\langle u, v \rangle} \coloneqq \frac{1}{4}\bigl(\mu^{\#}_{\langle u+v \rangle} - \mu^{\#}_{\langle u-v \rangle}\bigr)$, where $\mu^{\#}_{\langle \cdot \rangle} \in \bigl\{\mu_{\langle \cdot \rangle}, \mu^{c}_{\langle \cdot \rangle}, \mu^{j}_{\langle \cdot \rangle}, \mu^{k}_{\langle \cdot \rangle}\bigr\}$.

The following lemma is a Fatou-type property for energy measures.

Lemma A.11. *Let $\varphi\colon X \to [0,\infty)$ be Borel measurable with $\|\varphi\|_{\sup} < \infty$ and let $\mu^{\#}_{\langle\cdot\rangle} \in \{\mu_{\langle\cdot\rangle}, \mu^{c}_{\langle\cdot\rangle}, \mu^{j}_{\langle\cdot\rangle}, \mu^{k}_{\langle\cdot\rangle}\}$. If $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ and $u \in \mathcal{F}_e$ satisfy $\lim_{n\to\infty} u_n = u$ m-a.e. and $\sup_{n\in\mathbb{N}} \mathcal{E}(u_n, u_n) < \infty$, then*

$$\int_X \varphi\, d\mu^{\#}_{\langle u\rangle} \le \liminf_{n\to\infty} \int_X \varphi\, d\mu^{\#}_{\langle u_n\rangle}. \tag{A.8}$$

Proof. By extracting a subsequence of $\{u_n\}_n$ if necessary, we can assume that the limit $\lim_{n\to\infty}\int_X \varphi\, d\mu^{\#}_{\langle u_n\rangle}$ exists. By using a version [CF, Theorem A.4.1-(ii)] of the Banach–Saks theorem, we can find a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that $\{v_l\}_{l\in\mathbb{N}} \subseteq \mathcal{F}$ defined by $v_l := l^{-1}\sum_{k=1}^{l} u_{n_k}$ satisfies $\lim_{k\wedge l\to\infty}\mathcal{E}(v_k - v_l, v_k - v_l) = 0$. Noting that $\lim_{l\to\infty} v_l = u$ m-a.e. and using Proposition A.3, we have $\lim_{l\to\infty}\mathcal{E}(u - v_l, u - v_l) = 0$. Hence $\lim_{l\to\infty}\int_X \varphi\, d\mu^{\#}_{\langle v_l\rangle} = \int_X \varphi\, d\mu^{\#}_{\langle u\rangle}$ by Proposition A.9. By the triangle inequality for $\left(\int_X \varphi\, d\mu^{\#}_{\langle\cdot\rangle}\right)^{1/2}$,

$$\left(\int_X \varphi\, d\mu^{\#}_{\langle v_l\rangle}\right)^{1/2} \le \frac{1}{l}\sum_{k=1}^{l}\left(\int_X \varphi\, d\mu^{\#}_{\langle u_{n_k}\rangle}\right)^{1/2},$$

which implies (A.8) by letting $l \to \infty$. □

Now we can show that the integrals with respect to energy measures give 2-energy forms satisfying $(\mathrm{GC})_2$.

Proposition A.12. *Let $\varphi\colon X \to [0,\infty)$ be Borel measurable with $\|\varphi\|_{\sup} < \infty$ and let $\mu^{\#}_{\langle\cdot\rangle} \in \{\mu_{\langle\cdot\rangle}, \mu^{c}_{\langle\cdot\rangle}, \mu^{j}_{\langle\cdot\rangle}, \mu^{k}_{\langle\cdot\rangle}\}$. Then $(\int_X \varphi\, d\mu^{\#}_{\langle\cdot\rangle}, \mathcal{F}_e)$ is a 2-energy form on (X,m) satisfying $(\mathrm{GC})_2$.*

Proof. Let us fix $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,2]$, $q_2 \in [2,\infty]$ and $T = (T_1,\dots,T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfying (2.1) with 2 in place of p. It suffices to prove that for any $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in (\mathcal{F}\cap C_c(X))^{n_1}$ and any $\varphi \in \mathcal{F}\cap C_c(X)$,

$$\left\|\left(\left(\int_X \varphi\, d\mu^{\#}_{\langle T_l(\boldsymbol{u})\rangle}\right)^{1/2}\right)_{l=1}^{n_2}\right\|_{\ell^{q_2}} \le \left\|\left(\left(\int_X \varphi\, d\mu^{\#}_{\langle u_k\rangle}\right)^{1/2}\right)_{k=1}^{n_1}\right\|_{\ell^{q_1}}. \tag{A.9}$$

Indeed, we can extend (A.9) to any $\boldsymbol{u} \in \mathcal{F}_e^{n_1}$ and any Borel measurable function $\varphi\colon X \to [0,\infty]$ as follows. Let us start with the case $\varphi = \mathbb{1}_A$, where $A \in \mathcal{B}(X)$. By [Rud, Theorem 2.18], there exist sequences $\{K_n\}_{n\in\mathbb{N}}$ and $\{U_n\}_{n\in\mathbb{N}}$ such that $K_n \subseteq A \subseteq U_n$, K_n is compact, U_n is open and $\lim_{n\to\infty}\max_{v\in\{T_l(\boldsymbol{u})\}_l\cup\{u_k\}_k}\mu^{\#}_{\langle v\rangle}(U_n \setminus K_n) = 0$. By Urysohn's lemma, we can pick $\varphi_n \in C_c(X)$ so that $0 \le \varphi_n \le 1$, $\varphi_n|_{K_n} = 1$ and $\mathrm{supp}_X[\varphi_n] \subseteq U_n$. Applying (A.9) for φ_n, we obtain $\left\|\left(\mu^{\#}_{\langle T_l(\boldsymbol{u})\rangle}(K_n)^{1/2}\right)_{l=1}^{n_2}\right\|_{\ell^{q_2}} \le \left\|\left(\mu^{\#}_{\langle u_k\rangle}(U_n)^{1/2}\right)_{k=1}^{n_1}\right\|_{\ell^{q_1}}$. By letting $n \to \infty$, we get (A.9) with $\varphi = \mathbb{1}_A$, i.e.,

$$\left\|\left(\mu^{\#}_{\langle T_l(\boldsymbol{u})\rangle}(A)^{1/2}\right)_{l=1}^{n_2}\right\|_{\ell^{q_2}} \le \left\|\left(\mu^{\#}_{\langle u_k\rangle}(A)^{1/2}\right)_{k=1}^{n_1}\right\|_{\ell^{q_1}}. \tag{A.10}$$

By the reverse Minkowski inequality on $\ell^{q_1/2}$ and the Minkowski inequality on $\ell^{q_2/2}$ (see also (2.20)), we can extend (A.10) to (A.9) for any non-negative Borel measurable simple function φ on X, By the monotone convergence theorem, (A.9) holds for any Borel measurable function $\varphi\colon X \to [0,\infty]$. Next we will extend (A.9) to $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in \mathcal{F}_e^{n_1}$. Since $\mathcal{F} \cap C_c(X)$ is dense in $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$, there exists an approximating sequence $\{u_{k,n}\}_{n\in\mathbb{N}} \subseteq \mathcal{F} \cap C_c(X)$ for u_k for each $k \in \{1,\dots,n_1\}$. Set $\boldsymbol{u}_n \coloneqq (u_{1,n},\dots,u_{n_1,n})$. Then, for each $l \in \{1,\dots,n_2\}$, $\lim_{n\to\infty} T_l(\boldsymbol{u}_n) = T_l(\boldsymbol{u})$ m-a.e., $T_l(\boldsymbol{u}_n) \in \mathcal{F}$ and $\sup_{n\in\mathbb{N}} \mathcal{E}(T_l(\boldsymbol{u}_n), T_l(\boldsymbol{u}_n)) < \infty$ by Proposition A.2. Hence $T_l(\boldsymbol{u}) \in \mathcal{F}_e$ by Proposition A.5, and

$$\begin{aligned}
\left\| \left(\left(\int_X \varphi\, d\mu^{\#}_{\langle T_l(\boldsymbol{u})\rangle} \right)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} &\le \left\| \left(\left(\liminf_{n\to\infty} \int_X \varphi\, d\mu^{\#}_{\langle T_l(\boldsymbol{u}_n)\rangle} \right)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\
&\le \liminf_{n\to\infty} \left\| \left(\left(\int_X \varphi\, d\mu^{\#}_{\langle T_l(\boldsymbol{u}_n)\rangle} \right)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\
&\overset{\text{(A.9)}}{\le} \liminf_{n\to\infty} \left\| \left(\left(\int_X \varphi\, d\mu^{\#}_{\langle u_{k,n}\rangle} \right)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}} \\
&= \left\| \left(\left(\int_X \varphi\, d\mu^{\#}_{\langle u_k\rangle} \right)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}},
\end{aligned}$$

where we used Lemma A.11 in the first inequality and Proposition A.9 in the last equality. This implies that $(\int_X \varphi\, d\mu^{\#}_{\langle\cdot\rangle}, \mathcal{F}_e)$ is a 2-energy form on (X,m) satisfying (GC)_2.

Let us go back to the proof of (A.9) in the case $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in (\mathcal{F}\cap C_c(X))^{n_1}$ and $\varphi \in \mathcal{F} \cap C_c(X)$. Fix a metric d on X which is compatible with the given topology of X, an increasing sequence of relatively open sets $\{G_l\}_{l\in\mathbb{N}}$ with $\bigcup_{l\in\mathbb{N}} G_l = X$ and a sequence of positive numbers $\{\delta_l\}_{l\in\mathbb{N}}$ with $\delta_l \downarrow 0$ as $l \to \infty$. Then there exist a sequence of positive numbers $\{\beta_n\}_{n\in\mathbb{N}}$ with $\beta_n \uparrow \infty$ as $n \to \infty$, a family of positive Radon measures $\{\sigma_\beta\}_{\beta>0}$ on $X \times X$ and a family of positive Radon measures $\{m_\beta\}_{\beta>0}$ on X with $m_\beta \ll m$ such that for any $v \in \mathcal{F} \cap C_c(X)$,

$$\int_X \varphi\, d\mu_{\langle v\rangle} = \lim_{\beta\to\infty} \left(\frac{\beta}{2} \int_{X\times X} |v(x)-v(y)|^2 \varphi(x)\, \sigma_\beta(dx,dy) + \frac{\beta}{2} \int_X |v(x)|^2 \varphi(x)\, m_\beta(dx) \right), \tag{A.11}$$

and

$$\int_X \varphi\, d\mu^c_{\langle v\rangle} = \lim_{l\to\infty} \lim_{n\to\infty} \frac{\beta_n}{2} \int_{\{(x,y)\in G_l\times G_l \mid d(x,y)<\delta_l\}} |v(x)-v(y)|^2 \varphi(x)\, \sigma_{\beta_n}(dx,dy). \tag{A.12}$$

See [FOT, the equations just before (3.2.13) and (3.2.19)] for details. Note that $T_l(\boldsymbol{u}) \in \mathcal{F} \cap C_c(X)$ for each $l \in \{1,\dots,n_2\}$ by Proposition A.2 and $T_l(0) = 0$. If $q_2 < \infty$, then we have from (A.11) that

$$\sum_{l=1}^{n_2} \left(\int_X \varphi\, d\mu_{\langle T_l(\boldsymbol{u})\rangle} \right)^{q_2/2}$$

$$
\begin{aligned}
&\overset{(2.19)}{\leq} \lim_{\beta\to\infty}\biggl(\frac{\beta}{2}\int_{X\times X}\|T(\boldsymbol{u}(x))-T(\boldsymbol{u}(y))\|_{\ell^{q_2}}^2\,\varphi(x)\,\sigma_\beta(dx,dy)\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\frac{\beta}{2}\int_X\|T(\boldsymbol{u}(x))\|_{\ell^{q_2}}^2\,\varphi(x)\,m_\beta(dx)\biggr)^{q_2/2}\\
&\overset{(2.1)}{\leq} \lim_{\beta\to\infty}\left(\frac{\beta}{2}\int_{X\times X}\|\boldsymbol{u}(x)-\boldsymbol{u}(y)\|_{\ell^{q_1}}^2\,\varphi(x)\,\sigma_\beta(dx,dy)+\frac{\beta}{2}\int_X\|\boldsymbol{u}(x)\|_{\ell^{q_1}}^2\,\varphi(x)\,m_\beta(dx)\right)^{q_2/2}\\
&\overset{(*)}{\leq} \lim_{\beta\to\infty}\Biggl(\sum_{k=1}^{n_1}\biggl[\frac{\beta}{2}\int_{X\times X}|u_k(x)-u_k(y)|^2\,\varphi(x)\,\sigma_\beta(dx,dy)\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\frac{\beta}{2}\int_X|u_k(x)|^2\,\varphi(x)\,m_\beta(dx)\biggr]^{q_1/2}\Biggr)^{\frac{2}{q_1}\cdot\frac{q_2}{2}}\\
&= \left(\sum_{k=1}^{n_1}\left(\int_X\varphi\,d\mu_{\langle u_k\rangle}\right)^{q_1/2}\right)^{q_2/q_1},
\end{aligned}
$$

where we used the triangle inequality for a suitable L^{2/q_1}-norm on $(X\times X)\sqcup X$. Here $\sqcup$ denotes the disjoint union. The case $q_2=\infty$ is similar, so we obtain the desired estimate (A.9) for $\mu^{\#}_{\langle\cdot\rangle}=\mu_{\langle\cdot\rangle}$. The other case $\mu^{\#}_{\langle\cdot\rangle}\in\{\mu^{c}_{\langle\cdot\rangle},\mu^{j}_{\langle\cdot\rangle},\mu^{k}_{\langle\cdot\rangle}\}$ can be shown in a similar way by virtue of the expression in [FOT, (3.2.23)]. □

Next we see that "$|\nabla u|$" also satisfies similar contraction properties. To present the precise definition of the density, we recall the notion of *minimal energy dominant measure*.

Definition A.13 (Minimal energy dominant measure; [Hin10, Definition 2.1])**.** A σ-finite Borel measure μ on X is called a *minimal energy-dominant measure* of $(\mathcal{E},\mathcal{F})$ if and only if the following two conditions hold.

(i) For any $f\in\mathcal{F}$, we have $\mu_{\langle f\rangle}\ll\mu$.

(ii) If another σ-finite Borel measure μ' on X satisfies (i) with μ in place of μ', then $\mu\ll\mu'$.

The existence of minimal energy-dominant measure is proved in [Nak85, Lemma 2.2] (see also [Hin10, Lemma 2.3]). For any minimal energy-dominant measure μ of $(\mathcal{E},\mathcal{F})$, the same argument as in [Hin10, Proof of Lemma 2.2] implies that $\mu_{\langle f\rangle}\ll\mu$ for any $f\in\mathcal{F}_e$. In addition, for $\mu^{\#}_{\langle\cdot\rangle}\in\{\mu_{\langle\cdot\rangle},\mu^{c}_{\langle\cdot\rangle},\mu^{j}_{\langle\cdot\rangle},\mu^{k}_{\langle\cdot\rangle}\}$, we easily see that $\mu^{\#}_{\langle f,g\rangle}\ll\mu$ for any $f,g\in\mathcal{F}_e$. We define $\Gamma^{\#}_{\mu}(u,v)\coloneqq\frac{d\mu^{\#}_{\langle u,v\rangle}}{d\mu}$ and $\Gamma^{\#}_{\mu}(u)\coloneqq\Gamma^{\#}_{\mu}(u,u)$ for $u,v\in\mathcal{F}_e$.

Proposition A.14. *Let μ be a minimal energy-dominant measure of $(\mathcal{E},\mathcal{F})$ and for each $f\in\mathcal{F}_e$, let $\Gamma_\mu(f)\coloneqq d\mu_{\langle f\rangle}/d\mu$ and $\Gamma^{w}_{\mu}(f)\coloneqq d\mu^{w}_{\langle f\rangle}/d\mu$ for each $w\in\{c,j,k\}$. Let $\Gamma^{\#}_{\mu}(\,\cdot\,)\in\{\Gamma_\mu(\,\cdot\,),\Gamma^{c}_{\mu}(\,\cdot\,),\Gamma^{j}_{\mu}(\,\cdot\,),\Gamma^{k}_{\mu}(\,\cdot\,)\}$. Then for any $n_1,n_2\in\mathbb{N}$, $q_1\in(0,2]$, $q_2\in[2,\infty]$ and $T=(T_1,\dots,T_{n_2})\colon\mathbb{R}^{n_1}\to\mathbb{R}^{n_2}$ satisfying (2.1) with 2 in place of p and any $\boldsymbol{u}=(u_1,\dots,u_{n_1})\in\mathcal{F}_e^{n_1}$,*

$$\left\|\left(\Gamma^{\#}_{\mu}(T_l(\boldsymbol{u}))(x)^{1/2}\right)_{l=1}^{n_2}\right\|_{\ell^{q_2}}\leq\left\|\left(\Gamma^{\#}_{\mu}(u_k)(x)^{1/2}\right)_{k=1}^{n_1}\right\|_{\ell^{q_1}}\quad\text{for } \mu\text{-a.e. } x\in X. \tag{A.13}$$

Moreover, for any $p \in [q_1, q_2] \cap (0,\infty)$,

$$\left\| \left(\left(\int_X \Gamma_\mu^{\#}(T_l(\boldsymbol{u}))^{\frac{p}{2}}\, d\mu \right)^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \left(\left(\int_X \Gamma_\mu^{\#}(u_k)^{\frac{p}{2}}\, d\mu \right)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \tag{A.14}$$

Proof. We first establish a good μ-version of $\Gamma_\mu^{\#}(v)$ for each $v \in \mathcal{F}_e$. Fix $\{X_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}(X)$ such that $X_n \subseteq X_{n+1}$, $X = \bigcup_{n\in\mathbb{N}} X_n$ and $\mu(X_n) \in (0,\infty)$ for each $n \in \mathbb{N}$. Let $\{A_k\}_{k\in\mathbb{N}}$ be a countable open base for the topology of X. Set $A_k^0 := X \setminus A_k$ and $A_k^1 := A_k$ for each $k \in \mathbb{N}$, and define

$$\mathcal{A}_k := \left\{ \bigcup_{\alpha\in\mathcal{I}} A_k^\alpha \;\middle|\; \mathcal{I} \subseteq \{0,1\}^k \right\}, \quad k \in \mathbb{N},$$

where $A_k^\alpha := \bigcap_{i=1}^k A_k^{\alpha_i}$ for $\alpha = (\alpha_i)_{i=1}^k \in \{0,1\}^k$. Note that $\bigcup_{\alpha\in\mathcal{I}} A_k^\alpha = \emptyset$ if $\mathcal{I} = \emptyset$. Then $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ is a non-decreasing sequence of σ-algebras on X with $\bigcup_{k\in\mathbb{N}} \mathcal{A}_k$ generating $\mathcal{B}(X)$. Note that $\bigcup_{\alpha\in\{0,1\}^k} A_k^\alpha = X$ and that $A_k^\alpha \cap A_k^\beta = \emptyset$ for $\alpha, \beta \in \{0,1\}^k$ with $\alpha \neq \beta$. For $v \in \mathcal{F}_e$, $n, k \in \mathbb{N}$, $\alpha \in \{0,1\}^k$, define $\Gamma_\mu^{\#}(v)_{n,k} \colon X \to [0,\infty)$ by, for $x \in A_k^\alpha$,

$$\Gamma_\mu^{\#}(v)_{n,k}(x) := \begin{cases} \mu(A_k^\alpha \cap X_n)^{-1} \mu_{\langle v\rangle}^{\#}(A_k^\alpha \cap X_n) & \text{if } \mu(A_k^\alpha \cap X_n) > 0, \\ 0 & \text{if } \mu(A_k^\alpha \cap X_n) = 0. \end{cases} \tag{A.15}$$

We also set $\mu_n := \mu(X_n)^{-1}\mu((\cdot)\cap X_n)$ and $v_n^{\#} := \frac{d\mu_{\langle v\rangle}^{\#}((\cdot)\cap X_n)}{\mu((\cdot)\cap X_n)}$. Then we easily see that $\mathbb{E}_{\mu_n}[v_n^{\#} \,|\, \mathcal{A}_k] = \Gamma_\mu^{\#}(v)_{n,k}$ μ-a.e. on X_n and hence $\lim_{k\to\infty} \Gamma_\mu^{\#}(v)_{n,k} = v_n^{\#}$ μ-a.e. on X_n by the martingale convergence theorem (see, e.g., [Dud, Theorem 10.5.1]) and the fact that $\bigcup_{k\in\mathbb{N}} \mathcal{A}_k$ generates $\mathcal{B}(X)$. Now we define $\widetilde{\Gamma}_\mu^{\#}(v) \colon X \to [0,\infty)$ by $\widetilde{\Gamma}_\mu^{\#}(v)(x) := v_n^{\#}(x)$ for $n \in \mathbb{N}$ and $x \in X_n \setminus X_{n-1}$, where $X_0 := \emptyset$. Then $\widetilde{\Gamma}_\mu^{\#}(v) = \Gamma_\mu^{\#}(v)$ μ-a.e. on X.

Next we show (A.13). Let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,2]$, $q_2 \in [2,\infty]$, $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in \mathcal{F}_e^{n_1}$ and let $T = (T_1,\dots,T_{n_2}) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy (2.1) with 2 in place of p. By Proposition A.12 and (A.15), for any $n, m \in \mathbb{N}$ and any $x \in X$,

$$\left\| \left(\Gamma_\mu^{\#}(T_l(\boldsymbol{u}))_{n,m}(x)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \left(\Gamma_\mu^{\#}(u_k)_{n,m}(x)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}.$$

By letting $m \to \infty$, we obtain

$$\left\| \left(\widetilde{\Gamma}_\mu^{\#}(T_l(\boldsymbol{u}))(x)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \left(\widetilde{\Gamma}_\mu^{\#}(u_k)(x)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}} \quad \text{for } \mu\text{-a.e. } x \in X,$$

whence (A.13) holds. If $p \in [q_1, q_2] \cap (0,\infty)$ and $q_2 < \infty$, then we see that

$$\begin{aligned} \sum_{l=1}^{n_2} \left(\int_X \Gamma_\mu^{\#}(T_l(\boldsymbol{u}))^{\frac{p}{2}}\, d\mu \right)^{q_2/p} &\overset{(2.19)}{\le} \left(\int_X \left\| \left(\Gamma_\mu^{\#}(T_l(\boldsymbol{u}))(x)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}}^p \mu(dx) \right)^{q_2/p} \\ &\overset{(A.13)}{\le} \left(\int_X \left\| \left(\Gamma_\mu^{\#}(u_k)(x)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}^p \mu(dx) \right)^{q_2/p} \end{aligned}$$

$$\overset{(*)}{\leq} \left(\sum_{k=1}^{n_1} \left(\int_X \Gamma_\mu^{\#}(u_k)^{\frac{p}{2}} \, d\mu \right)^{q_1/p} \right)^{q_2/q_1}, \tag{A.16}$$

where we used the triangle inequality for the norm of $\ell_{n_1}^{p/q_1}$. The case $q_2 = \infty$ is similar, so we obtain (A.14). □

If $(\mathcal{E}, \mathcal{F})$ is strongly local, then we can show $(\mathrm{GC})_p$ for $(\Gamma_\mu(\,\cdot\,)^{p/2}, \mathcal{F}_e)$. To prove it, we need some preparations. The following proposition is the standard Minkowski integral inequality (see, e.g., [DF, Appendix B5]).

Proposition A.15. *Let $(X_i, \mathcal{B}_i, m_i)$ be a σ-finite measure space for each $i \in \{1, 2\}$. Let $q \in (1, \infty)$ and $f \colon X_1 \times X_2 \to \mathbb{R}$ be measurable. Then*

$$\left(\int_{X_1} \left(\int_{X_2} f(x_1, x_2) \, m_2(dx_2) \right)^q m_1(dx_1) \right)^{\frac{1}{q}} \leq \int_{X_2} \left(\int_{X_1} |f(x_1, x_2)|^q \, m_1(dx_1) \right)^{\frac{1}{q}} m_2(dx_2). \tag{A.17}$$

Next we show a tensor-type inequality for a bilinear form.

Proposition A.16. *Let V be a finite-dimensional vector space over $\mathbb{R}$, $E \colon V \times V \to \mathbb{R}$ a non-negative definite symmetric bilinear form, $n_1, n_2 \in \mathbb{N}$ and $A = (A_{lk})_{1 \leq l \leq n_2, 1 \leq k \leq n_1}$ a real matrix. Then for any $(u_1, \ldots, u_{n_1}) \in V^{n_1}$ and any $q_1 \in (0, \infty)$, $q_2 \in (0, \infty]$ with $q_1 \leq q_2$,*

$$\left\| \left(E\left(\sum_{k=1}^{n_1} A_{lk} u_k \right)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \leq \|A\|_{\ell_{n_1}^{q_1} \to \ell_{n_2}^{q_2}} \left\| \left(E(u_k)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}, \tag{A.18}$$

where we set $E(u) := E(u, u)$ for $u \in V$.

Proof. The desired inequality follows from a Beckner-like result in [DF, 7.9.] (see also [Bec75, Lemma 2]). We present a complete proof for convenience. Let γ_n be the Gaussian measure on $\mathbb{R}^n$, i.e., $\gamma_n(dx) := (2\pi)^{-n/2} \exp\left(-\|x\|^2 / 2 \right) dx$, for each $n \in \mathbb{N}$ and set $n := \dim(V/E^{-1}(0)) \in \mathbb{N} \cup \{0\}$. If $n = 0$, i.e., $E(u) = 0$ for any $u \in V$, then (A.18) is clear. Hence we assume that $n \geq 1$ in the rest of the proof. Let $\pi_j \colon \mathbb{R}^n \to \mathbb{R}$ be the projection map to the j-th coordinate for each $j \in \{1, \ldots, n\}$. Then we have from [DF, Proposition in 8.7.] that for any $(\alpha_j)_{j=1}^n \in \mathbb{R}^n$,

$$\|\pi_1\|_{L^{q_1}(\mathbb{R}, \gamma_1)}^{-1} \left(\int_{\mathbb{R}^n} \left| \sum_{j=1}^n \alpha_j \pi_j(x) \right|^{q_1} d\gamma_n(dx) \right)^{1/q_1} = \left\| (\alpha_j)_{j=1}^n \right\|_{\ell^2}. \tag{A.19}$$

Indeed, (A.19) is obviously true in the case $(\alpha_j)_j = (\delta_{1j})_j$ and this together with the invariance of γ_n under ℓ_n^2-isometries implies the desired equality (A.19).

Let us fix a basis $\{e_j\}_{j=1}^n \subseteq V$ of V satisfying $E(e_j, e_{j'}) = \delta_{jj'}$ for each $j, j' \in \{1, \dots, n\}$, which exists by the Gram–Schmidt orthonormalization. Now we define $\iota \colon V \to L^{q_1}(\mathbb{R}^n, \gamma_n)$ by

$$\iota(u) \coloneqq \|\pi_1\|_{L^{q_1}(\mathbb{R},\gamma_1)}^{-1} \sum_{j=1}^{n} E(u, e_j)^{1/2}\pi_j, \quad u \in V. \tag{A.20}$$

Then $\|\iota(u)\|_{L^{q_1}(\mathbb{R}^n,\gamma_n)} = \bigl(\sum_{j=1}^n E(u, e_j)\bigr)^{1/2} = E(u, u)^{1/2}$ by (A.19). If $q_2 < \infty$, then we see that

$$\begin{aligned}
\left\| \left(E\left(\sum_{k=1}^{n_1} A_{lk}u_k \right)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} &= \left(\sum_{l=1}^{n_2} \left(\int_{\mathbb{R}^n} \left| \sum_{k=1}^{n_1} A_{lk}\iota(u_k) \right|^{q_1} d\gamma_n \right)^{q_2/q_1} \right)^{\frac{q_1}{q_2}\cdot\frac{1}{q_1}} \\
&\overset{(*)}{\leq} \left(\int_{\mathbb{R}^n} \left(\sum_{l=1}^{n_2} \left| \sum_{k=1}^{n_1} A_{lk}\iota(u_k) \right|^{q_2} \right)^{q_1/q_2} d\gamma_n \right)^{1/q_1} \\
&\leq \|A\|_{\ell^{q_1}_{n_1} \to \ell^{q_2}_{n_2}} \left(\int_{\mathbb{R}^n} \sum_{k=1}^{n_1} |\iota(u_k)|^{q_1} \, d\gamma_n \right)^{1/q_1} \\
&= \|A\|_{\ell^{q_1}_{n_1} \to \ell^{q_2}_{n_2}} \left(\sum_{k=1}^{n_1} E(u_k)^{q_1/2} \right)^{1/q_1},
\end{aligned}$$

where we used (A.17) with $q = q_1/q_2$ in $(*)$. Since the case $q_2 = \infty$ is similar, so we obtain (A.18). $\square$

Let us recall the definition of p-energy forms introduced by Kuwae in [Kuw24]

Definition A.17 ([Kuw24, Definition 1.4]). Let μ be a minimal energy-dominant measure of $(\mathcal{E}, \mathcal{F})$, $p \in (1, \infty)$ and $\mathscr{D} \subseteq \{u \in L^p(X, m) \cap \mathcal{F} \mid \Gamma_\mu(u)^{\frac{1}{2}} \in L^p(X, \mu)\}$ a linear subspace. Assume that $(\mathcal{E}, \mathcal{F})$ is strongly local and that

$$\begin{aligned}&\lim_{n\to\infty} \int_X \Gamma_\mu(u_n)^{\frac{p}{2}} \, d\mu = 0 \text{ for any } \{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{D} \text{ with} \\ &\lim_{n\wedge m\to\infty} \int_X \Gamma_\mu(u_n - u_m)^{\frac{p}{2}} \, d\mu = 0 \text{ and } \lim_{n\to\infty} \|u_n\|_{L^p(X,m)} = 0.\end{aligned} \tag{A.21}$$

We define the norm $\|\cdot\|_{H^{1,p}}$ on $\mathscr{D}$ by $\|u\|_{H^{1,p}} \coloneqq \bigl(\|u\|_{L^p(X,m)}^p + \int_X \Gamma_\mu(u)^{\frac{p}{2}} \, d\mu\bigr)^{1/p}$ for $u \in \mathscr{D}$ and $H^{1,p}(X) \coloneqq \overline{\mathscr{D}}^{\|\cdot\|_{H^{1,p}}}$. Then, for $u \in H^{1,p}(X)$, we can uniquely extend Γ_μ to u with $\Gamma_\mu(u)^{\frac{1}{2}} \in L^p(X, \mu)$ as the $L^p(X, \mu)$-limit of $\Gamma_\mu(u_n)^{\frac{1}{2}}$, where $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{D}$ satisfies $\lim_{n\wedge m\to\infty} \int_X \Gamma_\mu(u_n - u_m)^{\frac{p}{2}} \, d\mu = 0$ and $\lim_{n\to\infty} \|u - u_n\|_{L^p(X,m)} = 0$.

Remark A.18. The condition (A.21) is verified in the case $p \geq 2$ in [Kuw24, Proposition 1.1].

Now we can show the main result in this subsection.

Theorem A.19. *Let μ be a minimal energy-dominant measure of $(\mathcal{E},\mathcal{F})$, $p \in (1,\infty)$ and $\mathscr{D} \subseteq \{u \in L^p(X,m) \cap \mathcal{F} \mid \Gamma_\mu(u)^{\frac{1}{2}} \in L^p(X,\mu)\}$ a linear subspace. Assume that $(\mathcal{E},\mathcal{F})$ is strongly local and that (A.21) holds. In addition, we assume that*

$$\begin{aligned}&\widehat{T}(u) \in \mathscr{D} \text{ for any } u \in \mathscr{D}^n \text{ and any } \widehat{T} \in C^\infty(\mathbb{R}^n) \text{ satisfying}\\ &\sup_{x,y\in\mathbb{R}^n; x\neq y} \frac{\bigl|\widehat{T}(x)-\widehat{T}(y)\bigr|}{\|x-y\|} < \infty \text{ and } \widehat{T}(0)=0.\end{aligned} \tag{A.22}$$

Then for any $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,p]$, $q_2 \in [p,\infty]$ and $T = (T_1,\dots,T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfying (2.1) and any $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in H^{1,p}(X)^{n_1}$, we have $T(\boldsymbol{u}) \in H^{1,p}(X)^{n_2}$ and

$$\bigl\|\bigl(\Gamma_\mu(T_l(\boldsymbol{u}))(x)^{1/2}\bigr)_{l=1}^{n_2}\bigr\|_{\ell^{q_2}} \le \bigl\|\bigl(\Gamma_\mu(u_k)(x)^{1/2}\bigr)_{k=1}^{n_1}\bigr\|_{\ell^{q_1}} \quad \text{for } \mu\text{-a.e. } x \in X. \tag{A.23}$$

In particular, $\bigl(\int_X \Gamma_\mu(\,\cdot\,)^{\frac{p}{2}}\,d\mu, H^{1,p}(X)\bigr)$ satisfies $(\mathrm{GC})_p$.

Proof. Let us consider the same mollifiers as in [Kuw24, The last paragraph in p. 10], i.e., define $j\colon \mathbb{R}^{n_1} \to \mathbb{R}$ by $j(x) \coloneqq \exp\bigl(-\frac{1}{1-\|x\|^2}\bigr)$ for $\|x\| \le 1$ and $j(x) \coloneqq 0$ for $\|x\| > 1$, set $j_m(x) \coloneqq m^{n_1} j(mx)$ for each $m \in \mathbb{N}$. We define $T_{l,n}(x) \coloneqq \int_{\mathbb{R}^{n_1}} (j_n(x-y)-j_n(y))T_l(y)\,dy = \int_{\mathbb{R}^{n_1}} j_n(y)(T_{l,n}(x-y)-T_{l,n}(y))\,dy$ so that $T_{l,n} \in C^\infty(\mathbb{R}^{n_1})$, $T_{l,n}(0) = 0$ and $\lim_{n\to\infty} T_{l,n}(x) = T_l(x)$ for any $x \in \mathbb{R}^{n_1}$. Then (2.1) with $T^{(n)} \coloneqq (T_{1,n},\dots,T_{n_2,n})$ in place of T holds; indeed, for any $x, y \in \mathbb{R}^{n_1}$,

$$\begin{aligned}\bigl\|T^{(n)}(x) - T^{(n)}(y)\bigr\|_{\ell^{q_2}} &= \left\|\left(\int_{\mathbb{R}^{n_1}} j_n(z)(T_l(x-z) - T_l(y-z))\,dz\right)_{l=1}^{n_2}\right\|_{\ell^{q_2}}\\ &\overset{(*)}{\le} \int_{\mathbb{R}^{n_1}} j_n(z)\,\|T(x-z) - T(y-z)\|_{\ell^{q_2}}\,dz\\ &\overset{(2.1)}{\le} \|x-y\|_{\ell^{q_1}} \int_{\mathbb{R}^{n_1}} j_n(z)\,dz = \|x-y\|_{\ell^{q_1}},\end{aligned} \tag{A.24}$$

where we used (A.17) with $q = q_2$ in $(*)$. Moreover,

$$\left\|\left(\sum_{k=1}^{n_1} \partial_k T_{l,n}(x) y_k\right)_{l=1}^{n_2}\right\|_{\ell^{q_2}} = \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\bigl\|T^{(n)}(x) - T^{(n)}(x+\varepsilon y)\bigr\|_{\ell^{q_2}} \overset{(A.24)}{\le} \|y\|_{\ell^{q_1}}, \tag{A.25}$$

whence $\|(\partial_k T_{l,n}(x))\|_{\ell^{q_1}_{n_1}\to\ell^{q_2}_{n_2}} \le 1$ for any $x \in \mathbb{R}^{n_1}$.

We first prove (A.23) with $T^{(n)}$ in place of T under the assumption that $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in \mathscr{D}^{n_1}$. Set $\widetilde{\boldsymbol{u}} = (\widetilde{u}_1,\dots,\widetilde{u}_{n_1})$ where $\widetilde{u}_k$ is a $\mathcal{E}$-quasicontinuous m-version of u_k (see [FOT, p. 69 and Theorem 2.1.3]). We have $T_{l,n}(\boldsymbol{u}) \in \mathscr{D}$ by (A.22) and

$$\Gamma_\mu(T_{l,n}(\boldsymbol{u}))(x) = \sum_{i,j=1}^{n_1} \partial_i T_{l,n}(\widetilde{\boldsymbol{u}}(x))\partial_j T_{l,n}(\widetilde{\boldsymbol{u}}(x))\Gamma_\mu(u_i,u_j)(x) \quad \text{for } \mu\text{-a.e. } x \in X \tag{A.26}$$

by the chain rule in [Kuw24, (7) in p. 2]. Let $\{f_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{F}$ be an algebraic basis of $\mathcal{F}$ over $\mathbb{R}$. Then there exist $n \in \mathbb{N}$, $\{\alpha_{k,j}\}_{j=1}^n \subseteq \mathbb{R}$, $k \in \{1,\dots,n_1\}$, and $\{g_j\}_{j=1}^n \subseteq \{f_\lambda\}_{\lambda\in\Lambda}$ such that $u_k = \sum_{j=1}^{n} \alpha_{k,j} g_j$ for each $k \in \{1,\dots,n_1\}$. Let R be the finitely generated algebra over $\mathbb{Q}$ generated by $\{\alpha_{k,j}\}_{1\le j\le n, 1\le k\le n_1} \cup \{1\}$ so that $\mathbb{Q} \subseteq R$ and R is countable. We set

$$\mathcal{U} \coloneqq \left\{ \sum_{j=1}^{n} a_j g_j \,\middle|\, a_j \in R \text{ for each } j \in \{1,\dots,n\} \right\}$$

so that $\{u_k\}_{k=1}^{n_1} \subseteq \mathcal{U}$ and $\mathcal{U}$ is countable. Since R is dense in $\mathbb{R}$, for any $x \in X$, $N \in \mathbb{N}$, $k \in \{1,\dots,n_1\}$ and $l \in \{1,\dots,n_2\}$, there exists $A_{lk,n}^{x,N} \in R$ such that $\left|\partial_k T_{l,n}(\widetilde{\boldsymbol{u}}(x)) - A_{lk,n}^{x,N}\right| \le N^{-1}$. Note that $\Gamma_\mu(\,\cdot\,,\,\cdot\,)(x) \colon \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ is a non-negative definite symmetric bilinear form for μ-a.e. $x \in X$ since $\mathcal{U}$ is countable. By Proposition A.16, for μ-a.e. $x \in X$,

$$\begin{aligned}
&\left\| \left(\left(\sum_{i,j=1}^{n_1} A_{li,n}^{x,N} A_{lj,n}^{x,N} \Gamma_\mu(u_i,u_j)(x) \right)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\
&= \left\| \left(\Gamma_\mu\left(\sum_{k=1}^{n_1} A_{lk,n}^{x,N} u_k \right)(x)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\
&\le \left(1 + \left\| (\partial_k T_{l,n}(\widetilde{\boldsymbol{u}}(x)))_{l,k} - (A_{lk,n}^{x,N})_{l,k} \right\|_{\ell_{n_1}^{q_1} \to \ell_{n_2}^{q_2}} \right) \left\| \left(\Gamma_\mu(u_k)(x)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}.
\end{aligned}$$

Letting $N \to \infty$ in the estimate above and recalling (A.26), we obtain

$$\left\| \left(\Gamma_\mu(T_{l,n}(\boldsymbol{u}))(x)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \left(\Gamma_\mu(u_k)(x)^{1/2} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}} \quad \text{for } \mu\text{-a.e. } x \in X, \tag{A.27}$$

under the assumption that $\boldsymbol{u} \in \mathscr{D}^{n_1}$.

Next let $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in H^{1,p}(X)^{n_1}$ and fix $\{\boldsymbol{u}^{(n)} = (u_{1,n},\dots,u_{n_1,n})\}_{n\in\mathbb{N}} \subseteq \mathscr{D}^{n_1}$ so that $\lim_{n\to\infty} \max_{k\in\{1,\dots,n_1\}} \|u_k - u_{k,n}\|_{H^{1,p}} = 0$. Then (A.27) together with the same same argument as in (A.16) implies that

$$\left\| \left(\left(\int_X \Gamma_\mu(T_{l,n}(\boldsymbol{u}^{(n)}))^{\frac{p}{2}}\, d\mu \right)^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \left(\left(\int_X \Gamma_\mu(u_{k,n})^{\frac{p}{2}}\, d\mu \right)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}.$$

In particular, $\{T_{l,n}(\boldsymbol{u}^{(n)})\}_{n\in\mathbb{N}}$ is bounded in $H^{1,p}(X)$. Noting that $H^{1,p}(X)$ is reflexive (see [Kuw24, Theorem 1.7]) and that $\lim_{n\to\infty} \int_X \Gamma_\mu(u_k - u_{k,n})^{\frac{p}{2}}\, d\mu = 0$, we find $\{n_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N}$ with $\inf_{j\in\mathbb{N}}(n_{j+1} - n_j) \ge 1$ such that $T^{(n_j)}(\boldsymbol{u}^{(n_j)})$ converges weakly in $H^{1,p}(X)^{\oplus n_2}$[13] to some $v = (v_1,\dots,v_{n_2}) \in H^{1,p}(X)^{\oplus n_2}$ and $\max_{k\in\{1,\dots,n_1\}} \Gamma_\mu(u_k - u_{k,n_j})(x) \to 0$ for μ-a.e. $x \in X$ as $j \to \infty$. Since $\lim_{n\to\infty} \left\| T_{l,n}(\boldsymbol{u}^{(n)}) - T_l(\boldsymbol{u}) \right\|_{L^p(X,m)} = 0$ by (A.24) and the dominated convergence theorem, we have $v_l = T_l(\boldsymbol{u})$. By Mazur's lemma (Lemma 3.13), there exist $\{N(i)\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$ and $\{\alpha_j\} \subseteq [0,1]$ with $\inf_{i\in\mathbb{N}}(N(i) - i) \ge 1$ and $\sum_{j=i}^{N(i)} \alpha_{i,j} = 1$ such that $\widehat{v}_{l,i} \coloneqq \sum_{j=i}^{N(i)} \alpha_{i,j} T_{l,n_j}(\boldsymbol{u}^{(n_j)})$ converges strongly in $H^{1,p}(X)$ to $T_l(\boldsymbol{u})$ for any $l \in \{1,\dots,n_2\}$

[13] The direct sum $H^{1,p}(X)^{\oplus n_2}$ is equipped with the norm $\|f\|_{H^{1,p}(X)^{\oplus n_2}} \coloneqq \sum_{l=1}^{n_2} \|f_j\|_{H^{1,p}(X)}$ for any $f = (f_1,\dots,f_{n_2}) \in H^{1,p}(X)^{\oplus n_2}$.

as $i \to \infty$. Then we easily see that for μ-a.e. $x \in X$ and any $i \in \mathbb{N}$,

$$\begin{aligned}\left\| \left(\Gamma_{\mu}(\widehat{v}_{l,i})(x)^{1/2}\right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} &\le \left\| \left(\sum_{j=i}^{N(i)} \alpha_{i,j} \Gamma_{\mu}(T_{l,n_j}(\boldsymbol{u}^{(n_j)}))(x)^{1/2} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\ &\le \sum_{j=i}^{N(i)} \alpha_{i,j} \left\| \left(\Gamma_{\mu}(T_{l,n_j}(\boldsymbol{u}^{(n_j)}))(x)^{1/2}\right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\ &\overset{\text{(A.27)}}{\le} \sum_{j=i}^{N(i)} \alpha_{i,j} \left\| \left(\Gamma_{\mu}(u_{k,n_j})(x)^{1/2}\right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}, \end{aligned} \tag{A.28}$$

where we used the triangle inequality for the norm of ℓ^{q_2} in the second inequality. Note that for μ-a.e. $x \in X$,

$$\lim_{i\to\infty} \sum_{j=i}^{N(i)} \alpha_{i,j} \left\| \left(\Gamma_{\mu}(u_{k,n_j})(x)^{1/2}\right)_{k=1}^{n_1} \right\|_{\ell^{q_1}} = \left\| \left(\Gamma_{\mu}(u_{k})(x)^{1/2}\right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}.$$

Since $\lim_{i\to\infty} \int_X \Gamma_{\mu}(\widehat{v}_{l,i} - T_l(\boldsymbol{u}))^{\frac{p}{2}}\,d\mu = 0$, there exists $\{m_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$ with $\inf_{i\in\mathbb{N}}(m_{i+1} - m_i) \ge 1$ such that $\lim_{i\to\infty} \Gamma_{\mu}(\widehat{v}_{l,m_i} - T_l(\boldsymbol{u}))(x) = 0$ for μ-a.e. $x \in X$ and any $l \in \{1, \dots, n_2\}$. In view of the triangle inequality for $\Gamma_{\mu}(\,\cdot\,)^{\frac{1}{2}}$ (see [Kuw24, (3) in p. 2]), we have $\lim_{i\to\infty} \max_{l\in\{1,\dots,n_2\}} \left|\Gamma_{\mu}(\widehat{v}_{l,m_i})(x) - \Gamma_{\mu}(T_l(\boldsymbol{u}))(x)\right| = 0$ for μ-a.e. $x \in X$. Hence we obtain (A.23) by (A.28). Once we get (A.23), we easily see that $(\int_X \Gamma_{\mu}(\,\cdot\,)^{\frac{p}{2}}\,d\mu, H^{1,p}(X))$ satisfies (GC)$_p$ by the same argument as in (A.16). □

B Some results for p-resistance forms on p.-c.f. self-similar structures

B.1 Existence of p-resistance forms with non-arithmetic weights

In this subsection, we discuss a gap between the frameworks in Subsection 8.2 and in Subsection 8.3 for p.-c.f. self-similar structures. As in Subsection 8.3, we fix $p \in (1, \infty)$ and a p.-c.f. self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ with $\#S \ge 2$ and K connected.

The following proposition about the "eigenvalue" $\lambda(\boldsymbol{\rho}_p)$ in Theorem 8.37 is a key result.

Proposition B.1. *Let $\boldsymbol{\rho}_p = (\rho_{p,i})_{i\in S} \in (0,\infty)^S$. Assume that $\boldsymbol{\rho}_p$ satisfies* (**A**) *(recall Remark 8.38).*

(a) *For any $a \in (0,\infty)$, $a\boldsymbol{\rho}_p \coloneqq (a\rho_{p,i})_{i\in S}$ satisfies* (**A**) *and $\lambda(a\boldsymbol{\rho}_p) = a\lambda(\boldsymbol{\rho}_p)$.*

(b) *Let $\widetilde{\boldsymbol{\rho}}_p = (\widetilde{\rho}_{p,i})_{i\in S} \in (0,\infty)^S$. If $\widetilde{\boldsymbol{\rho}}_p$ satisfies* (**A**) *and $\rho_{p,i} \le \widetilde{\rho}_{p,i}$ for any $i \in S$, then $\lambda(\boldsymbol{\rho}_p) \le \lambda(\widetilde{\boldsymbol{\rho}}_p)$.*

Proof. Throughout this proof, we fix a p-resistance form E_0 on V_0.

(a): Since $\mathcal{R}^n_{a\boldsymbol{\rho}_p}(E_0) = a\mathcal{R}^n_{\boldsymbol{\rho}_p}(E_0)$ for any $n \in \mathbb{N} \cup \{0\}$, we easily see that $a\boldsymbol{\rho}_p$ satisfies (**A**). Recall from Theorem 8.37-(a) that $\lambda(a\boldsymbol{\rho}_p) \in (0,\infty)$ is the unique number satisfying the following: there exists $C \in [1,\infty)$ such that

$$C^{-1}\lambda(a\boldsymbol{\rho}_p)^n E_0(u) \le \mathcal{R}^n_{a\boldsymbol{\rho}_p}(E_0)(u) \le C\lambda(a\boldsymbol{\rho}_p)^n E_0(u) \quad \text{for any } n \in \mathbb{N}\cup\{0\},\ u \in \mathbb{R}^{V_0}. \tag{B.1}$$

Therefore, $\lambda(a\boldsymbol{\rho}_p) = a\lambda(\boldsymbol{\rho}_p)$.

(b): Since $\mathcal{R}^n_{\boldsymbol{\rho}_p}(E_0)(u) \le \mathcal{R}^n_{\widetilde{\boldsymbol{\rho}}_p}(E_0)(u)$ for any $u \in \mathbb{R}^{V_0}$, by (B.1), there exists $C \in [1,\infty)$ such that for any $n \in \mathbb{N}\cup\{0\}$ and any $u \in \mathbb{R}^{V_0}$,

$$C^{-1}\lambda(\boldsymbol{\rho}_p)^n E_0(u) \le \mathcal{R}^n_{\boldsymbol{\rho}_p}(E_0)(u) \le \mathcal{R}^n_{\widetilde{\boldsymbol{\rho}}_p}(E_0)(u) \le C\lambda(\widetilde{\boldsymbol{\rho}}_p)^n E_0(u).$$

Since $n \in \mathbb{N}\cup\{0\}$ is arbitrary and $E_0(u) > 0$ for $u \in \mathbb{R}^{V_0} \setminus \mathbb{R}\mathbb{1}_{V_0}$, we conclude that $\lambda(\boldsymbol{\rho}_p) \le \lambda(\widetilde{\boldsymbol{\rho}}_p)$. □

Now we can show the existence of p-resistance forms with non-arithmetic weights on some affine nested fractals as follows.

Proposition B.2. *Let $\mathcal{L}$ be an affine nested fractal. Assume that there exists $i \in S$ such that*

$$\bigcup_{g\in\mathcal{G}} g^{(1)}(i) \neq S. \tag{B.2}$$

Then there exists $\boldsymbol{\rho}_p = (\rho_{p,i})_{i\in S} \in (0,\infty)^S$ such that $\lambda(\boldsymbol{\rho}_p) = 1$, $\rho_{p,i} > 1$ for any $i \in S$, $\boldsymbol{\rho}_p$ satisfies (8.65) and

$$\frac{\log \rho_{p,i}}{\log \rho_{p,j}} \notin \mathbb{Q} \quad \textit{for some } i,j \in S. \tag{B.3}$$

In particular, there exists a self-similar p-resistance form $(\mathcal{E}_p, \mathcal{F}_p)$ on $\mathcal{L}$ with weight $\boldsymbol{\rho}_p$.

Remark B.3. (1) Any weight $\boldsymbol{\rho}_p = (\rho_{p,i})_{i\in S}$ of a p-energy form constructed in Theorem 8.29 must satisfy $\rho_{p,i} = \sigma_p^{n_i}$ for some $n_i \in \mathbb{N}$, where $\sigma_p \in (0,\infty)$ is the p-scaling factor. Hence constructions of self-similar p-energy forms with weight ρ_p which satisfies (B.3) are not covered by Theorem 8.29 (and by [Kig23, Theorem 4.6]).

(2) The condition (B.2) is not so restrictive. See Figure B.2 for self-similar sets satisfying this condition. In Figure B.1, we present examples of self-similar sets that do not satisfy (B.2).

Proof. Fix $i \in S$ and set $S_1 := \bigcup_{g\in\mathcal{G}} g^{(1)}(i)$ and $S_2 := S \setminus S_1$, which is non-empty by (B.2). For $t \in \mathbb{R}$, we define $\boldsymbol{\rho}_p(t) := (\rho_{p,s}(t))_{s\in S}$ by

$$\rho_{p,s}(t) := 1 + t\mathbb{1}_{S_2}(s) \quad \text{for } s \in S.$$

It is easy to see that $\rho_p(t)$ satisfies (8.65). Set $\lambda_p(t) := \lambda(\boldsymbol{\rho}_p(t))$ for simplicity. By Proposition B.1, for any $t \in \mathbb{R}$, any $\delta \in (0,\infty)$ and any $s \in S$,

$$(1 - t - \delta)\lambda_p(0) \le \lambda_p(t-\delta) \le \lambda_p(t) \le \lambda_p(t+\delta) \le (1+t+\delta)\lambda_p(0),$$

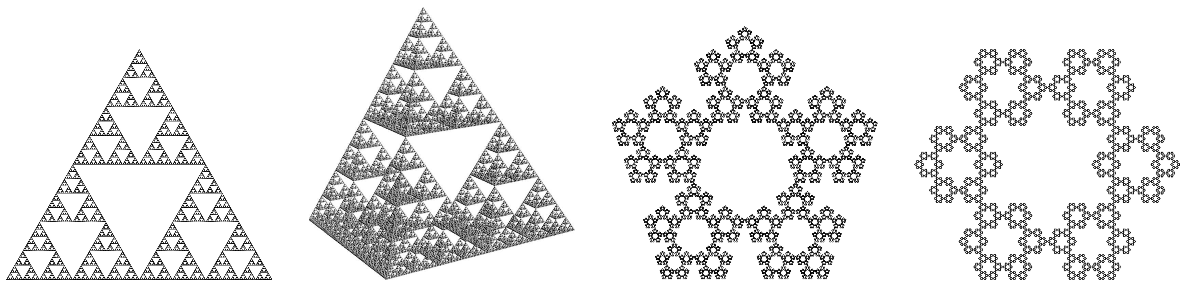

Figure B.1: Examples of affine nested fractals that do *NOT* satisfy (B.2). From the left, D-dimensional level-2 Sierpiński gasket ($D = 2, 3$), pentakun and hexagasket.

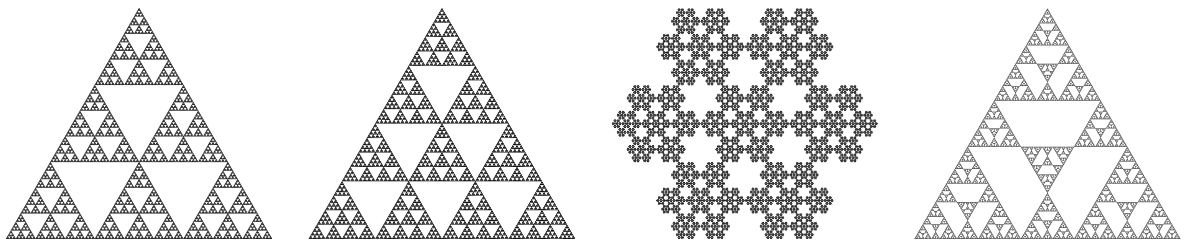

Figure B.2: Examples of affine nested fractals that satisfy (B.2). From the left, 2-dimensional level-l Sierpiński gasket ($l = 3, 4$), snowflake and a Sierpiński gasket-type fractal.

whence $\lambda_p(t)$ is continuous in t.

Fix $j \in S_2$ and define

$$r_{i,j}(t) := \frac{\log\left(\rho_{p,i}(t)/\lambda_p(t)\right)}{\log\left(\rho_{p,j}(t)/\lambda_p(t)\right)} = \frac{-\log\left(\lambda_p(t)\right)}{\log(1+t) - \log\left(\lambda_p(t)\right)}, \quad t \in \mathbb{R}.$$

Since $r_{i,j}(0) = 1$ and $r_{i,j}(t)$ is continuous in t, there exists $t_* \in \mathbb{R}\setminus\{0\}$ such that $r_{i,j}(t_*) \notin \mathbb{Q}$. The existence of a self-similar p-resistance form on $\mathcal{L}$ with weight $\boldsymbol{\rho}_p$ follows from Theorems 8.51 and 8.52, so we complete the proof. □

B.2 Ahlfors regular conformal dimension of affine nested fractals

In this subsection, we prove that the Ahlfors regular conformal dimension of any affine nested fractal equipped with the p-resistance metric for any $p \in (1, \infty)$ is 1. We also show that the Ahlfors regular conformal dimension with respect to the Euclidean metric is also 1 under some geometric condition,

Throughout this section, we assume that $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ is an affine nested fractal (see Framework 8.47 and Definition 8.48). Let $c_i \in (0, 1)$ be the contraction ratio of F_i for each $i \in S$. Note that $(c_i)_{i \in S} \in (0, 1)^S$ must satisfy

$$c_i = c_{g^{(1)}(i)} \quad \text{for any } i \in S \text{ and any } g \in \mathcal{G}, \tag{B.4}$$

because of the symmetry of $\mathcal{L}$. For each $p \in (1,\infty)$, we also fix a self-similar p-resistance form $(\mathcal{E}_p^{\#}, \mathcal{F}_p^{\#})$ on $\mathcal{L}$ with equal weight $(\rho_{\#,p})_{i\in S} \in (1,\infty)^S$ for some $\rho_{\#,p} \in (1,\infty)$, i.e., $\mathcal{F}_p^{\#} \subseteq C(K)$ and

$$\begin{aligned}\mathcal{F}_p^{\#} &= \{u \in C(K) \mid u \circ F_i \in \mathcal{F}_p^{\#} \text{ for any } i \in S\},\\ \mathcal{E}_p^{\#}(u) &= \rho_{\#,p} \sum_{i\in S} \mathcal{E}_p^{\#}(u \circ F_i) \quad \text{for any } u \in \mathcal{F}_p^{\#}.\end{aligned}$$

By Theorem 8.51, such a self-similar p-resistance form on $\mathcal{L}$ exists and the number $\rho_{\#,p}$ is uniquely determined. Let $\widehat{R}_p^{\#}$ denote the p-resistance metric associated with $(\mathcal{E}_p^{\#}, \mathcal{F}_p^{\#})$.

The following lemma describes good geometric properties of metric balls with respect to the p-resistance metrics. (The lemma below is true for p.-c.f. self-similar structures as well. See [KS.a, Section 6] for details.) Recall the definition of $U_M^{\widehat{R}_{p,\mathcal{E}_p}}(x,s)$ in Definition 7.10.

Lemma B.4. *Let $p \in (1,\infty)$ and let $(\mathcal{E}_p, \mathcal{F}_p)$ be a self-similar p-resistance form on $\mathcal{L}$ with weight $\rho_p = (\rho_{p,i})_{i\in S} \in (1,\infty)^S$.*

(a) *There exist $\alpha_1, \alpha_2 \in (0,\infty)$ such that for any $(s,x) \in (0,1] \times K$,*

$$B_{\widehat{R}_{p,\mathcal{E}_p}}(x, \alpha_1 s) \subseteq U_1^{\widehat{R}_{p,\mathcal{E}_p}}(x,s) \subseteq B_{\widehat{R}_{p,\mathcal{E}_p}}(x, \alpha_2 s). \tag{B.5}$$

(Equivalently, $\widehat{R}_{p,\mathcal{E}_p}$ is 1-adapted to the weight function $g(w) := \rho_{p,w}^{-1/(p-1)}$; see [Kig20, Definition 2.4.1].)

(b) *Let $d_{\mathrm{f}}(\boldsymbol{\rho}_p) \in (0,\infty)$ be such that $\sum_{i\in S} \rho_{p,i}^{-d_{\mathrm{f}}(\boldsymbol{\rho}_p)/(p-1)} = 1$, and let m be the self-similar measure on $\mathcal{L}$ with weight $\bigl(\rho_{p,i}^{-d_{\mathrm{f}}(\boldsymbol{\rho}_p)/(p-1)}\bigr)_{i\in S}$. Then there exist $c_1, c_2 \in (0,\infty)$ such that for any $(x,s) \in K \times (0, \operatorname{diam}(K, \widehat{R}_{p,\mathcal{E}_p})]$,*

$$c_1 s^{d_{\mathrm{f}}(\boldsymbol{\rho}_p)} \le m(B_{\widehat{R}_{p,\mathcal{E}_p}}(x,s)) \le c_2 s^{d_{\mathrm{f}}(\boldsymbol{\rho}_p)}. \tag{B.6}$$

In particular, $\widehat{R}_{p,\mathcal{E}_p}$ is metric doubling.

(c) *There exists $C \in (0,\infty)$ such that for any $(x,s) \in K \times (0, \operatorname{diam}(K, \widehat{R}_{p,\mathcal{E}_p})]$,*

$$\inf\Bigl\{\mathcal{E}_p(u) \Bigm| u \in \mathcal{F}_p, u|_{B_{\widehat{R}_{p,\mathcal{E}_p}}(x,\alpha_1 s)} = 1, \operatorname{supp}[u] \subseteq B_{\widehat{R}_{p,\mathcal{E}_p}}(x, 2\alpha_2 s)\Bigr\} \le C s^{-(p-1)}, \tag{B.7}$$

where α_1, α_2 are the constants in (B.5).

Proof. In this proof, we set $\widehat{R}_p := \widehat{R}_{p,\mathcal{E}_p}$ and $\Lambda_s := \Lambda_s^{\widehat{R}_p}$ for simplicity.

(a): By (7.1), we have $\operatorname{diam}(K_w, \widehat{R}_p) \le \rho_{p,w}^{-1/(p-1)} \operatorname{diam}(K, \widehat{R}_p)$ for any $w \in W_*$, which implies the latter inclusion in (B.5) with $\alpha_2 \in (2\operatorname{diam}(K, \widehat{R}_p), \infty)$ arbitrary. (In particular, $\operatorname{diam}(K_w, \widehat{R}_p) < \alpha_2 s$ for any $w \in \Lambda_s$.) We will show the former inclusion in (B.5). It suffices to prove that there exists $\alpha_1 \in (0,\infty)$ such that $\widehat{R}_p(x,y) \ge \alpha_1 s$ for any $s \in (0,1]$, any $w, v \in \Lambda_s$ with $K_w \cap K_v = \emptyset$ and any $(x, y) \in K_w \times K_v$. Let $\psi_q := h_{V_0}^{\mathcal{E}_p}[\mathbb{1}_q^{V_0}]$ for any $q \in V_0$. Fix $w \in \Lambda_s$ and let $u_w \in C(K)$ be such that, for $\tau \in \Lambda_s$,

$$u_w \circ F_\tau = \begin{cases} 1 & \text{if } \tau = w, \\ \sum_{q \in V_0; F_\tau(q) \in F_w(V_0)} \psi_q & \text{if } \tau \neq w \text{ and } K_\tau \cap K_w \neq \emptyset, \\ 0 & \text{if } K_\tau \cap K_w = \emptyset. \end{cases} \tag{B.8}$$

By the self-similarity for $(\mathcal{E}_p, \mathcal{F}_p)$, we have $u_w \in \mathcal{F}_p$ and

$$\mathcal{E}_p(u_w) = \sum_{\tau \in \Lambda_s} \rho_{p,\tau} \mathcal{E}_p(u_w \circ F_\tau) = \sum_{\tau \in \Lambda_s \setminus \{w\}; K_\tau \cap K_w \neq \emptyset} \rho_{p,\tau} \mathcal{E}_p \left(\sum_{q \in V_0; F_\tau(q) \in F_w(V_0)} \psi_q \right). \tag{B.9}$$

(Note that Λ_s is a partition of Σ.) Set $\overline{\rho}_p := \max_{i \in S} \rho_{p,i} \in (1, \infty)$ and $c_1 := \max_{q \in V_0} \mathcal{E}_p(\psi_q) \in (0, \infty)$. Then $\rho_{p,\tau}^{-1} \geq (\overline{\rho}_p)^{-1} s^{p-1}$ for any $\tau \in \Lambda_s$. Since $\#\{\tau \in \Lambda_s \mid K_\tau \cap K_w \neq \emptyset\} \leq (\#\mathcal{C}_\mathcal{L})(\#V_0)$ by [Kig01, Lemma 4.2.3], (B.9) together with Hölder's inequality implies that

$$\mathcal{E}_p(u_w) \leq (\#\mathcal{C}_\mathcal{L})(\#V_0) \overline{\rho}_p s^{-p+1} (\#V_0)^{p-1} c_1 =: (\alpha_1 s)^{-(p-1)}. \tag{B.10}$$

For any $v \in \Lambda_s$ with $K_w \cap K_v = \emptyset$ and any $(x, y) \in K_w \times K_v$, we clearly have $u_w(x) = 1$ and $u_w(y) = 0$. Hence

$$\widehat{R}_p(x, y) \geq \mathcal{E}_p(u)^{-1/(p-1)} \geq \alpha_1 s,$$

which proves the desired result.

(b): This is immediate from (B.5), $\#\{\tau \in \Lambda_s \mid K_\tau \cap K_w \neq \emptyset\} \leq (\#\mathcal{C}_\mathcal{L})(\#V_0)$ (see [Kig01, Lemma 4.2.3]) and $m(K_w) = \rho_{p,w}^{-1/(p-1)}$ (see [Kig01, Corollary 1.4.8]).

(c): Let $u_w \in \mathcal{F}_p$ be the same function as in the proof of (a) for each $w \in \Lambda_s$. Then $\varphi := \max_{w \in \Lambda_{s,1}(x)} u_w$ satisfies $\varphi|_{U_1(x,s)} = 1$. Since $\operatorname{diam}(K_w, \widehat{R}_p) < \alpha_2 s$, we see from (B.5) that $\operatorname{supp}[\varphi] \subseteq B_{\widehat{R}_p}(x, 2\alpha_2 s)$. By (2.5) for $(\mathcal{E}_p, \mathcal{F}_p)$, (B.10) and [Kig01, Lemma 4.2.3], we have $\varphi \in \mathcal{F}_p$ and

$$\mathcal{E}_p(\varphi) \leq \sum_{w \in \Lambda_{s,1}(x)} \mathcal{E}_p(u_w) \leq (\alpha_1 s)^{-(p-1)} (\#\mathcal{C}_\mathcal{L})(\#V_0) =: C s^{-(p-1)}. \qquad \square$$

The next proposition ensures that $\widehat{R}_p^{\#}$ is quasisymmetric to the q-resistance metric with respect to *any* self-similar q-resistance form arising from Theorem 8.51. (Recall Definition 8.5-(3).)

Proposition B.5. *Let $p, q \in (1, \infty)$ and assume that $\boldsymbol{\rho}_q = (\rho_{q,i})_{i \in S} \in (0, \infty)^S$ satisfies (8.65), $\rho_{q,i} > 1$ for any $i \in S$ and $\lambda(\boldsymbol{\rho}_q) = 1$, where $\lambda(\boldsymbol{\rho}_q) \in (0, \infty)$ is the unique number given in Theorem 8.51. Let $(\mathcal{E}_q, \mathcal{F}_q)$ be a self-similar q-resistance form on $\mathcal{L}$ with weight $\boldsymbol{\rho}_q$, which exists by Theorems 8.51, and let $\widehat{R}_q$ be the q-resistance metric associated with $(\mathcal{E}_q, \mathcal{F}_q)$. Then $\widehat{R}_{q,\mathcal{E}_q}$ is quasisymmetric to $\widehat{R}_p^{\#}$.*

Proof. We will use [Kig20, Corollary 3.6.7] to show the desired statement. We first show that there exist $\alpha_1, \alpha_2 \in (0,\infty)$ such that

$$\alpha_1 \rho_{q,w}^{-1/(p-1)} \leq \operatorname{diam}(K_w, \widehat{R}_q) \leq \alpha_2 \rho_{q,w}^{-1/(p-1)} \quad \text{for any } w \in W_*. \tag{B.11}$$

The upper estimate in (B.11) is immediate from (7.1). To prove the lower estimate in (B.11), note that we can easily find $m_0 \in \mathbb{N}$ such that for any $w \in W_*$ there exist $v^1, v^2 \in W_{|w|+m_0}$ with $v^i \leq w$, $i = 1,2$, and $K_{v^1} \cap K_{v^2} = \emptyset$. (It is enough to choose m_0 satisfying $2(\max_{i \in S} c_i)^{m_0} < 1$.) Then, by combining the proof of Lemma B.4-(a) and $\rho_{p,v^i} \leq \rho_{q,w}(\max_{i \in S} \rho_{q,i})^{m_0}$, there exists $\alpha_1 \in (0,\infty)$ that is independent of $w \in W_*$ such that

$$\inf_{(x,y) \in K_{v^1} \times K_{v^2}} \widehat{R}_q(x,y) \geq \alpha_1 \rho_{q,w}^{-1/(p-1)},$$

which implies the desired lower estimate in (B.11).

Next we note that $\mathcal{L}$ is a rationally ramified self-similar structure by [Kig09, Proposition 1.6.12]; moreover, by combining [Kig09, Proposition 1.6.12], $K_v \cap K_w = F_v(V_0) \cap F_w(V_0)$ for any $v, w \in W_*$ with $\Sigma_v \cap \Sigma_w = \emptyset$ (see [Kig01, Proposition 1.3.5-(2)]) and the fact that each element of V_0 is a fixed point of F_i for some $i \in S_{\mathrm{fix}} := \{i \in S \mid K_i \cap V_0 \neq \emptyset\}$, $\mathcal{L}$ is rationally ramified with a relation set

$$\mathcal{R} = \bigl\{\{(\{w(j)\}, \{v(j)\}, \varphi_j, x(j), y(j)) \mid w(j), v(j), x(j), y(j) \in W_* \setminus \{\emptyset\}\}\bigr\}_{j=1}^{k} \tag{B.12}$$

satisfying $w(j), v(j) \in S_{\mathrm{fix}}$. (See [Kig09, Sections 1.5 and 1.6 and Chapter 8] for details about rationally ramified self-similar structures.)

With these preparations, we will apply [Kig20, Corollary 3.6.7] for $\widehat{R}_{q,\mathcal{E}_q}$ and $\widehat{R}_p^{\#}$. By Lemma B.4-(a) and (B.11), $\widehat{R}_{q,\mathcal{E}_q}$ is 1-*adapted* and *exponential* (see [Kig20, Definition 2.4.7 and 3.1.15-(2)] for these definitions; see also Remark in [Kig20, p. 108]). Similarly, $\widehat{R}_p^{\#}$ is also 1-adapted and exponential. Hence, by [Kig20, Corollary 3.6.7], $\widehat{R}_{q,\mathcal{E}_q}$ is quasisymmetric to $\widehat{R}_p^{\#}$ if and only if $\widehat{R}_{q,\mathcal{E}_q}$ is *gentle* with respect to $\widehat{R}_p^{\#}$ (see [Kig20, Definition 3.3.1] for the definition of the gentleness). Define $g_q(w) := \rho_{q,w}^{-1/(q-1)}$ and $g_{\#,p}(w) := \rho_{\#,p}^{-|w|}$ for $w \in W_*$. Since g_q and $g_{\#,p}$ satisfy the condition (R1) in [Kig09, Theorem 1.6.6] by (8.65) and (B.12), we obtain the desired gentleness by [Kig09, Theorem 1.6.6] and (B.11). This completes the proof. □

Now we can determine the Ahlfors regular conformal dimension of $(K, \widehat{R}_p^{\#})$.

Theorem B.6. $\dim_{\mathrm{ARC}}(K, \widehat{R}_p^{\#}) = 1$.

Proof. We will use the characterization of the Ahlfors regular conformal dimension in [Kig20, Theorem 4.6.9]. Note that $(K, \widehat{R}_p^{\#})$ satisfies (BF1) and (BF2) in [Kig20, Section 4.3] by Lemma B.4-(a), (B.11), [Kig09, Proposition 1.6.12, Lemmas 1.3.6 and 1.3.12]. We define a graph $G_n = (V_n, E_n)$ and q-energy $\mathcal{E}_p^{G_n}$, $q \in (1,\infty)$, on G_n by

$$E_n := \{(x,y) \mid x, y \in F_w(V_0) \text{ for some } w \in W_n\},$$

and

$$\mathcal{E}_q^{G_n}(f) \coloneqq \frac{1}{2} \sum_{(x,y)\in E_n} |f(x) - f(y)|^q, \quad f \in \mathbb{R}^{V_n}.$$

Note that $\{G_n\}_{n\geq 0}$ is a *proper system of horizontal networks* with indices $(1, 2(\#V_0 - 1)\#V_0, 1, 1)$ (see [Kig20, Definition 4.6.5]). Hence, by [Kig20, Theorem 4.6.9], $\dim_{\mathrm{ARC}}(K, \widehat{R}_p^{\#}) = 1$ if and only if the following holds: for any $q \in (1, \infty)$,

$$\liminf_{k\to\infty} \sup_{w\in W_*} \inf\Bigl\{\mathcal{E}_q^{G_{|w|+k}}(f) \Bigm| f \in \mathbb{R}^{V_{|w|+k}}, f|_{F_w(V_k)} = 1, f|_{Z_{w,k}} = 0\Bigr\} = 0, \tag{B.13}$$

where $Z_{w,k} \coloneqq \{x \in V_{|w|+n} \mid x \in F_v(V_k) \text{ for some } v \in W_{|w|} \text{ with } K_v \cap K_w = \emptyset\}$. Since both $\mathcal{E}_q^{\#}\big|_{V_0}(\,\cdot\,)^{1/q}$ and $\mathcal{E}_q^{G_0}(\,\cdot\,)^{1/q}$ are norms on the finite-dimensional vector space $\mathbb{R}^{V_0}/\mathbb{R}\mathbb{1}_{V_0}$, there exists $C \geq 1$ such that $C^{-1}\mathcal{E}_q^{\#}\big|_{V_0}(u) \leq \mathcal{E}_q^{G_0}(u) \leq C\mathcal{E}_q^{\#}\big|_{V_0}(u)$ for any $u \in \mathbb{R}^{V_0}$. Hence, by Propositions 7.2-(3) and 7.4, we obtain $C^{-1}\mathcal{E}_q^{\#}\big|_{V_n}(u) \leq \rho_{\#,q}^{n}\mathcal{E}_q^{G_n}(u) \leq C\mathcal{E}_q^{\#}\big|_{V_n}(u)$ for any $n \in \mathbb{N} \cup \{0\}$ and any $u \in \mathbb{R}^{V_n}$. Recall that $\Gamma_1(w) = \{v \in W_{|w|} \mid K_v \cap K_w \neq \emptyset\}$ for $w \in W_*$ (see Definition 8.3). Let $h_{q,w} \in \mathcal{F}_q^{\#}$ be the unique function satisfying $h_{q,w}|_{K_w} = 1$, $h_{q,w}|_{K_v} = 0$ for any $v \in W_{|w|} \setminus \Gamma_1(w)$ and

$$\mathcal{E}_q^{\#}(h_{q,w}) = \inf\Bigl\{\mathcal{E}_q^{\#}(u) \Bigm| u|_{K_w} = 1, u|_{K_v} = 0 \text{ for any } v \in W_{|w|} \setminus \Gamma_1(w)\Bigr\}.$$

Then we see from (B.7), (B.5) and (B.11) that

$$\begin{aligned}
&\sup_{w\in W_*} \inf\Bigl\{\mathcal{E}_q^{G_{|w|+k}}(f) \Bigm| f \in \mathbb{R}^{V_{|w|+k}}, f|_{F_w(V_k)} = 1, f|_{Z_{w,k}} = 0\Bigr\} \\
&\leq C\rho_{\#,q}^{-(|w|+k)} \sup_{w\in W_*} \mathcal{E}_q^{\#}\big|_{V_{|w|+k}}(h_{q,w}|_{V_{|w|+k}}) \leq C\rho_{\#,q}^{-(|w|+k)} \sup_{w\in W_*} \mathcal{E}_q^{\#}(h_{q,w}) \lesssim \rho_{\#,q}^{-k}.
\end{aligned}$$

Since $\rho_{\#,q} \in (1, \infty)$ for any $q \in (0, 1)$, we obtain (B.13). The proof is completed. □

To discuss the Ahlfors regular conformal dimension of K with respect to the Euclidean metric, we need the following assumption.

Assumption B.7. We define $\Lambda_1^d \coloneqq \{\emptyset\}$,

$$\Lambda_s^d \coloneqq \{w \mid w = w_1 \dots w_n \in W_* \setminus \{\emptyset\}, \operatorname{diam}(K_{w_1\dots w_{n-1}}, d) > s \geq \operatorname{diam}(K_w, d)\}$$

for each $s \in (0, 1)$. For $s \in (0, 1]$, $M \in \mathbb{N} \cup \{0\}$ and $x \in K$, define

$$\Lambda_{s,M}^d(x) \coloneqq \left\{ v \;\middle|\; \begin{array}{l} v \in \Lambda_s^d, \text{ there exists } w \in \Lambda_s^d \text{ with } x \in K_w \text{ and} \\ \{z(j)\}_{j=1}^k \subseteq \Lambda_s^d \text{ with } k \leq M+1,\ z(1) = w,\ z(k) = v \\ \text{such that } K_{z(j)} \cap K_{z(j+1)} \neq \emptyset \text{ for any } j \in \{1, \dots, k-1\} \end{array} \right\},$$

and $U_M^d(x, s) \coloneqq \bigcup_{w\in\Lambda_{s,M}^d(x)} K_w$. Then there exist $M_* \in \mathbb{N}$, $\alpha_0, \alpha_1 \in (0, \infty)$ such that

$$U_{M_*}^d(x, \alpha_0 s) \subseteq B_d(x, s) \subseteq U_{M_*}^d(x, \alpha_1 s) \quad \text{for any } (x, s) \in K \times (0, 1].$$

(Equivalently, d is M_*-adapted; see [Kig20, Definition 2.4.1].)

Remark B.8. We do not know whether Assumption B.7 is true for any affine nested fractal. Even for a nested fractal, being 1-adapted with respect to the Euclidean metric is assumed in [Kig23, Assumption 4.41].

Now we can show the main result in this section under Assumption B.7.

Theorem B.9. *Assume that Assumption B.7 holds. Then* $\dim_{\mathrm{ARC}}(K, d) = 1$.

Proof. Thanks to Theorem B.6, it suffices to prove that $\widehat{R}_p^{\#}$ is quasisymmetric to d. Obviously, d is exponential since $\mathrm{diam}(K_w, d) = c_w \,\mathrm{diam}(K, d)$. By (B.4), a similar argument as in the proof of Proposition B.5 implies that $\widehat{R}_p^{\#}$ is gentle with respect to d. Hence [Kig20, Corollary 3.6.7] together with Assumption B.7 implies that $\widehat{R}_p^{\#}$ is quasisymmetric to d. □

B.3 An estimate on self-similar regular p-resistance forms on p.-c.f. self-similar structures

This subsection aims to prove the following result, which is a generalization of [Kig03, Theorem A.1].

Theorem B.10. *Let* $p \in (1, \infty)$ *and let* $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ *be a p.-c.f. self-similar structure with* $\#S \geq 2$. *Assume that there exists a self-similar p-resistance form* $(\mathcal{E}, \mathcal{F})$ *on* $\mathcal{L}$ *with weight* $\boldsymbol{\rho} = (\rho_i)_{i \in S}$ *and that* $\min_{i \in S} \rho_i > 1$. *Then there exists* $c \in (0, 1)$ *such that for any* $x, y \in K$ *and any* $w \in W_*$,

$$c\rho_w^{-1} R_{\mathcal{E}}(x, y) \leq R_{\mathcal{E}}(F_w(x), F_w(y)) \leq \rho_w^{-1} R_{\mathcal{E}}(x, y). \tag{B.14}$$

Since the upper estimate in (B.14) is obtained in (7.1), what matters is the lower estimate in (B.14). To prove it, we need the following lemma.

Lemma B.11. *Assume the same conditions as in Theorem B.10. Let* $x, y \in K$ *and* $w \in W_*$. *Set* $\Lambda := \{\tau = \tau_1 \ldots \tau_n \in W_* \mid (\rho_{\tau_1 \cdots \tau_{n-1}})^{-1} > \rho_w \geq \rho_\tau^{-1}\}$, $U := V_0 \cup \{x, y\}$, $V_\Lambda := \bigcup_{w \in \Lambda} F_w(V_0)$ *and* $V := V_\Lambda \cup \{F_w(x), F_w(y)\}$. *Then* Λ *is a partition of* Σ *and*

$$\mathcal{E}|_V(u) = \rho_w \mathcal{E}|_U(u \circ F_w) + \sum_{\tau \in \Lambda \setminus \{w\}} \rho_\tau \mathcal{E}|_{V_0}(u \circ F_\tau) \quad \textit{for any } u \in \mathcal{F}|_V. \tag{B.15}$$

Proof. The proof is very similar to Proposition 7.4. It is clear that Λ is a partition of Σ. Note that $C(K) = C(K, R_{\mathcal{E}}^{1/p})$ and $(K, R_{\mathcal{E}}^{1/p})$ is bounded by Proposition 7.2-(3). For any $u \in \mathcal{F}|_V$,

$$\begin{aligned}
&\mathcal{E}|_V(u) \\
&= \min\bigl\{\mathcal{E}(v) \bigm| v \in \mathcal{F}, v|_V = u\bigr\} \\
&\overset{(5.7)}{=} \min\left\{\rho_w \mathcal{E}(v \circ F_w) + \sum_{\tau \in \Lambda \setminus \{w\}} \rho_\tau \mathcal{E}(v \circ F_\tau) \,\middle|\, v \in \mathcal{F},\, v|_V = u\right\}
\end{aligned}$$

$$\geq \min\left\{\rho_w\mathcal{E}(v\circ F_w) \,\middle|\, v\in\mathcal{F}, v|_V = u\right\} + \min\left\{\sum_{\tau\in\Lambda\setminus\{w\}}\rho_\tau\mathcal{E}(v\circ F_\tau) \,\middle|\, v\in\mathcal{F}, v|_V = u\right\}$$

$$\geq \rho_w \min\{\mathcal{E}(v) \mid v\in\mathcal{F}, v|_U = u\circ F_w\} + \sum_{\tau\in\Lambda\setminus\{w\}}\rho_\tau \min\{\mathcal{E}(v) \mid v\in\mathcal{F}, v|_{V_0} = u\circ F_\tau\}$$

$$= \rho_w\mathcal{E}|_U(u\circ F_w) + \sum_{\tau\in\Lambda\setminus\{w\}}\rho_\tau\mathcal{E}|_{V_0}(u\circ F_\tau).$$

To prove the converse, let $v\in C(K)$ satisfy $v\circ F_w = h_U^{\mathcal{E}}[u\circ F_w]$ and, for $\tau\in\Lambda\setminus\{w\}$, $v\circ F_\tau = h_{V_0}^{\mathcal{E}}[u\circ F_\tau]$. Such v is well-defined since $K_w\cap K_\tau = F_w(V_0)\cap F_\tau(V_0)$. Also, we have $v|_V = u$ and $v\in\mathcal{F}$ by (5.5). Moreover,

$$\mathcal{E}|_V(u) \leq \mathcal{E}(v) \overset{(5.7)}{=} \sum_{\tau\in\Lambda}\rho_\tau\mathcal{E}(v\circ F_\tau) = \rho_w\mathcal{E}|_U(u\circ F_w) + \sum_{\tau\in\Lambda\setminus\{w\}}\rho_\tau\mathcal{E}|_{V_0}(u\circ F_\tau).$$

This completes the proof. □

Proof of Theorem B.10. Let Λ, U, V_Λ, V be the same as in Lemma B.11. Set $\Gamma_1(w;\Lambda) \coloneqq \{\tau\in\Lambda \mid w\neq\tau, K_w\cap K_\tau\neq\emptyset\}$ for simplicity. Then $\#\Gamma_1(w;\Lambda) \leq \#(\mathcal{C}_\mathcal{L})\#(V_0)$ by [Kig01, Lemma 4.2.3]. Let $\psi_{xy}\in\mathcal{F}$ satisfy $\psi_{xy}(x) = 1$, $\psi_{xy}(y) = 0$ and $\mathcal{E}(\psi_{xy}) = R_\mathcal{E}(x,y)^{-1}$. Let $u_*\in\mathcal{F}$ satisfy $u_*(x) = 1$, $u_*(y) = 0$, $u|_{V\setminus F_w(U)}\in\mathbb{R}\mathbb{1}_{V\setminus F_w(U)}$ and

$$\mathcal{E}(u_*) = \inf\{\mathcal{E}(v) \mid v\in\mathcal{F}, (v\circ F_w)|_U = \psi_{xy}, v|_{V\setminus F_w(U)}\in\mathbb{R}\mathbb{1}_{V\setminus F_w(U)}\}.$$

Such u_* is uniquely exists by a standard argument in the variational analysis. Also, by Proposition 2.2-(b), we easily see that $0\leq u_*\leq 1$. Since $\mathbb{R}^{V_0}/\mathbb{R}\mathbb{1}_{V_0}$ is a finite dimensional vector space, there exists a constant $C\in(0,\infty)$ such that

$$\mathcal{E}|_{V_0}(u)^{1/p} \leq C\max_{z,z'\in V_0}|u(z)-u(z')| \quad \text{for any } u\in\mathbb{R}^{V_0}. \tag{B.16}$$

Then, by using Lemma B.11, we see that

$$\begin{aligned}
R_\mathcal{E}(F_w(x),F_w(y))^{-1} \leq \mathcal{E}(u_*) &= \mathcal{E}|_V(u_*) \\
&= \rho_w\mathcal{E}|_U(u_*\circ F_w) + \sum_{\tau\in\Lambda\setminus\{w\}}\rho_\tau\mathcal{E}|_{V_0}(u_*\circ F_\tau) \\
&= \rho_w\mathcal{E}|_U(u_*\circ F_w) + \sum_{\tau\in\Gamma_1(w;\Lambda)}\rho_\tau\mathcal{E}|_{V_0}(u_*\circ F_\tau) \\
&\overset{(B.16)}{\leq} \frac{\rho_w}{R_\mathcal{E}(x,y)} + C^p\sum_{\tau\in\Gamma_1(w;\Lambda)}\rho_\tau \\
&\leq \rho_w\left(\frac{1}{R_\mathcal{E}(x,y)} + C^p\Bigl(\max_{i\in S}\rho_i\Bigr)(\#\Gamma_1(w;\Lambda))\right) \\
&= \rho_w\left(\frac{1}{R_\mathcal{E}(x,y)} + C'\frac{R_\mathcal{E}(x,y)}{R_\mathcal{E}(x,y)}\right) \\
&\leq \rho_w\Bigl(1 + C'\sup_{z,z'\in K}R_\mathcal{E}(z,z')\Bigr)R_\mathcal{E}(x,y)^{-1},
\end{aligned}$$

which shows the desired lower estimate in (B.14). □

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